FYS A 116 Integral transforms 17 XII 2013

Only writing materials (no handbook, calculator, computer, etc.).

1. Show how the exponential function e^x is created by means of Heaviside's differential and integral operator.

2. Justify: $f(x) = 1$ is a periodic function. What is its period? By applying this periodicity property find the Laplace transform of $f(x) = 1$.

3. Find $f(x)$ when $\mathcal{L} f(x) = \dfrac{1}{p^2 + (a+b)p + ab}$ $(a \neq b)$, using two different methods: (a) the convolution theorem, and (b) Heaviside's expansion.

4. A simple oscillator of mass m and spring constant k is subjected to an impulsive external force at $t = t_1 > 0$. Initially at $t = 0$ the oscillator is at rest at the origin. Using the Laplace transform find the position x of the oscillator as a function of time t when $t > t_1$.

5. The Fourier cosine series of $f(x) = x^2$ $(-\pi \le x \le \pi)$, $f(x+2\pi) = f(x)$, is $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$. Making use of this series determine (a) $\zeta(2)$, and (b) $\zeta(4)$. Here $\zeta(s)$ is the Riemann zeta function.

6. Derive d'Alembert's solution to the wave equation $\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0$ $(y = y(x,t))$, with initial conditions $y(x,0) = f(x)$ and $\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0$ $(-\infty < x < \infty)$, using the Fourier transform.

A table of basic formulas ("kaavakokoelma") ③

I. (a) $e^{\pm iax} = \cos(ax) \pm i\sin(ax)$

(b) $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (s>1)$.

II. (a) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\xi\, e^{ik(x-\xi)} f(\xi)$

(b) $$\left.\begin{aligned} \mathcal{F} f(x) &= \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx = F(k) \\ f(x) &= \mathcal{F}^{-1} F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} F(k)\,dk \end{aligned}\right\}$$

(c) $$\left.\begin{aligned} \mathcal{L} f(x) &= \int_{0}^{\infty} e^{-px} f(x)\,dx = F(p) \\ f(x) &= \mathcal{L}^{-1} F(p) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{px} F(p)\,dp \end{aligned}\right\}.$$

III. (a) $\mathcal{L}\, x = \frac{1}{p^2}$; $\mathcal{L}\, e^{ax} = \frac{1}{p-a}$; $\mathcal{L} \sin(ax) = \frac{a}{p^2+a^2}$;

$\mathcal{L}\cos(ax) = \frac{p}{p^2+a^2}$; $\mathcal{L}\sinh(ax) = \frac{a}{p^2-a^2}$; $\mathcal{L}\cosh(ax) = \frac{p}{p^2-a^2}$; $\mathcal{L}^{-1}\left[\frac{G(p)}{H(p)}\right] = \sum_{m} \frac{G(p_m)}{H'(p_m)} e^{p_m x}$

(b) $\mathcal{L} f^{(n)}(x) = p^n \mathcal{L} f(x) - \sum_{m=0}^{n-1} p^{n-m-1} f^{(m)}(0)$

(c) $\mathcal{L} \int_0^x f(\xi)\,d\xi = \frac{\mathcal{L} f(x)}{p}$

(d) $\mathcal{L}\left[f_1(x) * f_2(x)\right] = \mathcal{L} f_1(x) \cdot \mathcal{L} f_2(x)$;

$f_1(x) * f_2(x) = \int_0^x f_1(x-u) f_2(u)\,du$

(e) $\mathcal{L}\left[f(x-a)\,\theta(x-a)\right] = e^{-ap} F(p)$

$\mathcal{L}\left[e^{ax} f(x)\right] = F(p-a)$

(f) $\mathcal{L} f(x) = (1-e^{-pL})^{-1} \int_0^L e^{-px} f(x)\,dx \quad \left(f(x+L) = f(x)\right).$

IV. (a) $\mathcal{F} f^{(n)}(x) = (ik)^n \mathcal{F} f(x)$

(b) $\mathcal{F} \int_{-\infty}^{x} f(\xi)\, d\xi = \frac{\mathcal{F} f(x)}{ik}$

(c) $\mathcal{F}[f_1(x) * f_2(x)] = \mathcal{F} f_1(x) \cdot \mathcal{F} f_2(x)$;

$f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(x-u) f_2(u)\, du$

(d) $\int_{-\infty}^{\infty} [f(x)]^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(k)|^2 dk$

(e) $\mathcal{F} f(x-a) = e^{-ika} F(k)$

$\mathcal{F}[e^{iax} f(x)] = F(k-a)$.

V. (a) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x}$; $k_n = \frac{2\pi n}{L}$;

$c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-ik_n x} dx$

(b) $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos(k_n x) + b_n \sin(k_n x)]$;

$a_0 = \frac{2}{L} \int_{-L/2}^{L/2} f(x)\, dx$, $a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos(k_n x)\, dx$, $b_n =$
$\frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin(k_n x)\, dx$

(c) $\frac{1}{L} \int_{-L/2}^{L/2} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{4} |a_0|^2 + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 +$
$|b_n|^2)$.

VI. (a) $\int_a^b \delta(x-\xi) f(\xi)\, d\xi = f(x)$ $(a \le x \le b)$

(b) $\theta(x-c) = \begin{cases} 1 & (x > c) \\ 0 & (x < c) \end{cases}$;

$\frac{d\theta(x-c)}{dx} = \delta(x-c)$; $\theta(x) - \theta(-x) = \operatorname{sgn} x$

(c) $L(x)\, y(x) = r(x)$; $L(x)\, G(x, x') = \delta(x-x')$.