FYSA233, Quantum Mechanics I, part A, exam2 **22.11.2013**

The maximum points available in this exam is 48. Some useful formulas at the end.

1. (a) Explain what means "collapse of the wave function". [3 pt]

(b) Show that the eigenvalues of a self-adjoint operator are real valued. [3 pt]

(c) Find the normalization constant of the ground state wave function of 1D harmonic oscillator $\psi_0(x) = A\, \exp(-\frac{m\omega}{2\hbar}x^2)$. [3 pt]

(d) Explain what the Heisenberg uncertainty relation means. [3 pt]

yht. 12 pt

2. (a) Let the potential of a 1-dimensional quantum mechanical system be time-independent, that is the Hamiltonian operator is given by

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\,.$$

Use a separation of variables for the corresponding time-dependent Schrödinger equation to derive the stationary state Schrödinger equation $\hat{H}\psi_E(x) = E\psi_E(x)$ for this system, and solve the time-dependent part. Write down the general solution to the time-dependent Schrödinger equation in the basis spanned by the solutions of the corresponding stationary state Schrödinger equation. [4 pt]

(b) Start from the definition $\langle A(t)\rangle \equiv \langle\psi(t)|\hat{A}(t)|\psi(t)\rangle$ of the expectation value of an observable and show using Dirac's notation (without going into x-representation) that its time-evolution is governed by

$$\frac{d\langle A\rangle}{dt} = \left\langle\frac{\partial\hat{A}(t)}{\partial t}\right\rangle + \frac{\imath}{\hbar}\left\langle[\hat{H},\hat{A}]\right\rangle\,.$$

[4 pt]

(c) Show using the Heisenberg equation of motion derived above that the so-called Ehrenfest theorem, that is

$$\frac{d\langle x(t)\rangle}{dt} = \frac{\langle p(t)\rangle}{m}\,,$$
$$\frac{d\langle p(t)\rangle}{dt} = -\left\langle\frac{dV(x)}{dx}\right\rangle\,,$$

holds for the Hamiltonian operator of part (a). Notice that the result $[\hat{p},\hat{V}] = -\imath\hbar\frac{dV(x)}{dx}\Big|_{\hat{x}}$ might be helpful. [4 pt]

yht. 12 pt

3. (a) Let us examine the motion of a quantum mechanical particle that is scattered by a delta function potential at the origin $x=0$: $V(x) = \frac{\hbar^2}{m}\Omega\delta(x)$. The wave function of the particle is

$$\psi(x) = \begin{cases} A\,\exp(ikx) + B\,\exp(-ikx), & x<0 \\ C\,\exp(ikx), & x\geq 0 \end{cases}\,.$$

What kind of motion is represented by this wave function? [2 pt]

(b) Write the stationary Schrödinger equation for this problem and show that the discontinuity of the derivative of the wave function at origin is

$$\psi'(0_+) - \psi'(0_-) = 2\Omega\psi(0) \quad .$$

[5 pt]

(c) Although the derivative is discontinuous, the wave function itself is continuous everywhere. Use this information and derive the results for the reflectance and transmission coefficients

$$R = \frac{|B|^2}{|A|^2} = \frac{\Omega^2}{k^2+\Omega^2}$$
$$T = \frac{|C|^2}{|A|^2} = \frac{k^2}{k^2+\Omega^2}$$

[5 pt]

yht. 12 pt

4. Let us consider a two-state system with orthonormal states $|1\rangle$ ja $|2\rangle$ and energy eigenvalues E_1 ja E_2, which means

$$\hat{H}|1\rangle = E_1|1\rangle \, ,$$
$$\hat{H}|2\rangle = E_2|2\rangle \, .$$

Let there be also an observable A with a corresponding operator $\hat{A}$ that acts in the following way:

$$\hat{A}|1\rangle = +\imath\hbar|2\rangle \, ,$$
$$\hat{A}|2\rangle = -\imath\hbar|1\rangle \, .$$

Let the system be initially ($t = 0$) in state

$$|\psi(0)\rangle = \frac{3}{5}|1\rangle + \frac{4}{5}|2\rangle$$

Let the system evolve to time $t = t_1 > 0$.

(a) What are the possible eigenvalues for A and energy at time $t = t_1$? [4 pt]

(b) What are the probabilities to find the different eigenvalues of A at time $t = t_1$? [4 pt]

(c) Is A a constant of motion? Give arguments (calculation) to support your answer. [4 pt]

Some help:

$$|\psi(t)\rangle = \sum_N c_N e^{-\imath E_N(t-t_0)/\hbar}|E_N\rangle \, .$$

yht. 12 pt

Here is the list of some useful trigonometric identities

$$\begin{aligned}
1 &= \cos^2(x) + \sin^2(x)\,, \\
e^{\imath x} &= \cos(x) + \imath\sin(x)\,, \\
\cos(2x) &= \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)\,, \\
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y)\,, \\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y)\,,
\end{aligned}$$

integrals

$$\int_{-\infty}^{\infty} dx\, \exp(-bx^2) = \sqrt{\pi/b}$$

and commutation relations

$$[AB, C] = A[B, C] + [A, C]B\,.$$