FYSA235, Kvanttimekaniikka I, osa B, tentti2 **10.1.2014**

Tentin yhteispistemäärä on 48 pistettä. Kaavakokoelma ja CG-taulukko paperinipun lopussa.

1. Selitä lyhyesti

(a) Larmorin prekessio [3 pt]

(b) Clebsch-Gordan kertoimet [3 pt]

(c) bosonien ja fermionien aaltofunktioitten symmetriaominaisuudet hiukkasten vaihdossa [3 pt]

(d) Paulin kieltosääntö [3 pt]

yht. 12 pt

2. Tarkastellaan yhtä elektronia kytkemättömässä kvanttitilassa

$$|\ell = 1, s = 1/2, m_\ell = 0, m_s = 1/2\rangle_u.$$

Kytke pyörimismäärät ($\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$) ja laske mitä mahdollisia arvoja ja millä todennäkösyydellä systeemistä voidaan mitata kun siihen operoidaan samanaikaisesti operaattoreilla $\hat{\mathbf{J}}^2$ ja $\hat{J}_z$. **yht. 12 pt**

3. Tarkastellaan systeemiä jonka Hamiltonin operaattori on määritelty normitetussa ominaiskannassaan seuraavasti

$$\hat{H}_0|1\rangle = E_1^{(0)}|1\rangle, \quad \hat{H}_0|2\rangle = E_1^{(0)}|2\rangle, \quad \hat{H}_0|3\rangle = E_2^{(0)}|3\rangle, \quad \hat{H}_0|4\rangle = E_3^{(0)}|4\rangle \qquad (E_1^{(0)} < E_2^{(0)} < E_3^{(0)})$$

Lisätään systeemiin pieni häiriö ($g\epsilon > 0$ ja $|g\epsilon| << |E_1^{(0)}|, |E_2^{(0)}|, |E_3^{(0)}|$) siten että

$$g\hat{V}|1\rangle = g\epsilon(|1\rangle + |2\rangle), \quad g\hat{V}|2\rangle = g\epsilon|1\rangle, \quad g\hat{V}|3\rangle = -g\epsilon|3\rangle, \quad g\hat{V}|4\rangle = 3g\epsilon|4\rangle$$

(a) Muodosta häiriömatriisi $g\mathbf{V}$ $\hat{H}_0$:n ominaiskannassa. [4 pt]

(b) Laske 1. kertaluvun häiriöteoriaa käyttäen korjaukset energian ominaisarvoihin. [4 pt]

(c) Piirrä energiatasokaavio josta näkyy kuinka em. energiat muuttuvat häiriön seurauksena. [4 pt]

yht. 12 pt

4. Tarkastellaan yksiulotteista kvanttimekaanista systeemiä jonka Hamiltonin operaattori on muotoa

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

jossa V on deltapotentiaali $V(x) = -\alpha\delta(x), \quad \alpha > 0$. Arvioi perustilan energia tässä systeemissä käyttäen variaatiomenetelmää ja gaussista aaltofunktioyritettä $\psi(x) = Ae^{-bx^2}$ jossa A on normitusvakio ja $b > 0$ variaatioparametri (muista normittaa ensin). Vertaa saamaasi tulosta tarkkaan ratkaisuun $E_0 = -m\alpha^2/(2\hbar^2)$.

yht. 12 pt

FYSA235, Quantum mechanics I, part B, exam 2 **10.1.2014**

Total points 48. Useful formulas and the CG table attached at the end.

1. Explain briefly

(a) Larmor precession [3 pt]

(b) Clebsch-Gordan coefficients [3 pt]

(c) The symmetry properties of bosonic and fermionic wave functions regarding exchange of identical particles [3 pt]

(d) Pauli exclusion principle [3 pt]

yht. 12 pt

2. Let us consider one electron in an uncoupled quantum state

$$|\ell = 1, s = 1/2, m_\ell = 0, m_s = 1/2\rangle_u.$$

Make the coupling of angular momenta ($\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$) and determine what values and with what probabilities one can measure from the system when simultaneous measurements are done corresponding to operators $\hat{\mathbf{J}}^2$ ja $\hat{J}_z$.

yht. 12 pt

3. Let us consider a system whose Hamilton operator is defined in its egenbasis as

$$\hat{H}_0|1\rangle = E_1^{(0)}|1\rangle, \quad \hat{H}_0|2\rangle = E_1^{(0)}|2\rangle, \quad \hat{H}_0|3\rangle = E_2^{(0)}|3\rangle, \quad \hat{H}_0|4\rangle = E_3^{(0)}|4\rangle \qquad (E_1^{(0)} < E_2^{(0)} < E_3^{(0)})$$

The system is subjected to a weak perturbation ($g\epsilon > 0$ and $|g\epsilon| << |E_1^{(0)}|, |E_2^{(0)}|, |E_3^{(0)}|$) such that

$$g\hat{V}|1\rangle = g\epsilon(|1\rangle + |2\rangle), \quad g\hat{V}|2\rangle = g\epsilon|1\rangle, \quad g\hat{V}|3\rangle = -g\epsilon|3\rangle, \quad g\hat{V}|4\rangle = 3g\epsilon|4\rangle$$

(a) Write the perturbation matrix $g\mathbf{V}$ in the eigenbasis of $\hat{H}_0$. [4 pt]

(b) Evaluate the corrections to the energy eigenvalues by using the 1st order perturbation theory. [4 pt]

(c) Draw an energy level diagram that shows how the unperturbed energies are changed as the result of the perturbation. [4 pt]

yht. 12 pt

4. Let us consider a one-dimensional quantum system whose Hamilton operator is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

where V is a delta potential $V(x) = -\alpha\delta(x), \quad \alpha > 0$. Estimate the ground state energy of this system by using the variational principle and a Gaussian trial wave function $\psi(x) = Ae^{-bx^2}$ where A is the normalization constant and $b > 0$ variational parameter (remember to normalize). Compare your result to the known exact ground state energy $E_0 = -m\alpha^2/(2\hbar^2)$.

yht. 12 pt

Trigonometry

$$
\begin{aligned}
1 &= \cos^2(x) + \sin^2(x)\,, \\
e^{\imath x} &= \cos(x) + \imath\sin(x)\,, \\
\cos(2x) &= \cos^2(x) - \sin^2(x) = 2\cos^2(x) + 1 = 1 - 2\sin^2(x)\,, \\
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y)\,, \\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y)\,,
\end{aligned}
$$

Second degree polynomial equation

$$ax^2 + bx + c = 0 \quad \rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integrals

$$\int_0^\infty dx\ e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (a > 0)$$

$$\int_0^\infty dx\ x^{2n} e^{-ax^2} = \frac{(2n)!}{n! 2^{2n+1}}\sqrt{\frac{\pi}{a^{2n+1}}} \quad (a > 0, \quad n = 1, 2, 3, ...)$$

$$\int_0^\infty dx\ x^{2n+1} e^{-ax^2} = \frac{n!}{2a^{n+1}} \quad (a > 0, \quad n = 0, 1, 2, ...)$$

$$\int_0^\infty dx\ x^n e^{-ax} = \frac{n!}{a^{n+1}} \quad (n = 0, 1, 2, ...)$$

Commutator relations

$$[AB, C] = A[B, C] + [A, C]B\,.$$

Spherical coordinates and spherical functions

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{1}{\hbar^2 r^2}\hat{\mathbf{L}}^2 \qquad d^3r = r^2 dr d\Omega = r^2 dr sin\theta d\theta d\phi \qquad \int d\Omega = 4\pi$$

General angular momentum

$$[\hat{J}_i, \hat{J}_j] = \imath\hbar\sum_{k=1}^{3}\epsilon_{ijk}\hat{J}_k \qquad [\hat{\mathbf{J}}^2, \hat{J}_i] = 0 \qquad \hat{\mathbf{J}}^2|j,m\rangle = \hbar^2 j(j{+}1)|j,m\rangle \qquad \hat{J}_z|j,m\rangle = \hbar m|j,m\rangle$$

$$\hat{J}_\pm = \hat{J}_x \pm \imath\hat{J}_y \qquad \hat{J}_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$$

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. .			

1/2×1/2

m_1	m_2	1
		+1
+1/2	+1/2	1

m_1	m_2	1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1
		−1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2
		+3/2
+1	+1/2	1

m_1	m_2	3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2
		−3/2
−1	−1/2	1

2×1/2

m_1	m_2	5/2
		+5/2
+2	+1/2	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2
		−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2
		+2
+3/2	+1/2	1

m_1	m_2	2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2
		−2
−3/2	−1/2	1

2×1

m_1	m_2	3
		+3
+2	+1	1

m_1	m_2	3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3
		−3
−2	−1	1

1×1

m_1	m_2	2
		+2
+1	+1	1

m_1	m_2	2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2
		−2
−1	−1	1

3/2×1

m_1	m_2	5/2
		+5/2
+3/2	+1	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2
		−5/2
−3/2	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

3/2×3/2

m_1	m_2	3
		+3
+3/2	+3/2	1

m_1	m_2	3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3
		−3
−3/2	−3/2	1

2×3/2

m_1	m_2	7/2
		+7/2
+2	+3/2	1

m_1	m_2	7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2
		−7/2
−2	−3/2	1

2×2

m_1	m_2	4
		+4
+2	+2	1

m_1	m_2	4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	4
		−4
−2	−2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).