FYSA231 Quantum Mechanics I, part A 5.11.2010

1. exam: 4 problems, 3 hours

1. Consider quantum mechanical time evolution
   (a) (1p) Write the differential equation descibing quantum mechanical time evolution an explain shortly the notation you use.
   (b) (3p) Alterantively, quantum mechanical time evolution can be written as
   $$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle.$$
   Apply the equation you wrote in part (a), and derive the differential equation satisfied by the operator $U(t)$. What is the initial condition for $U(t)$ at $t = 0$.
   (c) (1p) Solve this equation by assuming that the Hamiltonian operator does not depend explicitly on time.
   (d) (1p) Let $|\phi(0)\rangle$ be a stationary state. How does this state evolve in time?

2. Consider Hamiltonian
   $$H = \begin{pmatrix} E_0 & 0 \\ 0 & -E_0 \end{pmatrix}$$
   in the eigenbasis of spin-1/2 operator S_z.
   (a) (1p) At time $t = 0$ the energy of the system is measured E_0. What is the state of the system immediately after the measurement?
   (b) (1p) If we measure the spin component along the x-axis from the sate obtained in the previous part, what are the possible results?
   (c) (4p) Suppose the the measurement of S_x gave $+\hbar/2$. Then the system is in the eigenstate corresponding to this result. Suppose that we wait time t without disturbing the system in any way. At time t the spin component S_z is measured. Calculate the probabilities to observe $s_z = \pm\hbar/2$.

3. (a) (3p) Show that the eigenvalues of a Hermitian operator are real and the eigenvectors are orthogonal. What is the significance of this mathematical result in quantum mechanics?
   (b) (1p) What is characteristic of an entangled state?
   (c) (2p) Consider a state vector
   $$|\Phi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2),$$
   where the vectors $|\pm\rangle_i$ ($i = 1, 2$) are the (orthogonal) eigenstates of spin-1/2 operator $S_z^{(i)}$,
   $$S_z^{(i)}|\pm\rangle_i = \pm\frac{\hbar}{2}|\pm\rangle_i.$$
   Show, that $|\Phi\rangle$ is an eigenstate of the operator $[S_z^{(1)} \otimes S_z^{(2)}]$. What is the corresponding eigenvalue?

Turn over!

4. Consider scattering from potential ($V_0 > 0$)

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geq 0. \end{cases}$$

and restrict to states with energy $E > V_0$.

(a) (2) Write the time independent Schrödinger equation in domains I: $x < 0$ and II: $x \geq 0$, and solve it.

(b) (2) Assume that there is a source of particles only at $x = -\infty$, and determine the coefficients in the solution from the continuity conditions of the wave function and its derivative.

(c) (2) Compute the reflection and transmission coefficients. What is the essential difference in comparison to classical scattering from the same potential?

Useful formulas:

$$e^{i\pi} + 1 = 0.$$

Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-x^2} = \sqrt{\pi}$$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Spin-1/2 spin operator:

$$\vec{S} = (S_x, S_y, S_z) = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z).$$

Spherical coordinates:

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta, \\ dxdydz &= r^2 \sin\theta dr d\theta d\phi. \end{aligned}$$

Continuity equation:

$$\frac{\partial \rho(t, \vec{x})}{\partial t} + \nabla \cdot \vec{j}(t, \vec{x}) = 0.$$