FYSA233 Quantum Mechanics I, Part A 23.11. 2012

final exam: 4 problems, 4 hours

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1. Consider quantum mechanical two-state system. An observable A has eigenstates $|+\rangle$ and $|-\rangle$, which satisfy

$$A|+\rangle = |+\rangle, \quad A|-\rangle = -|-\rangle.$$

The Hamiltonian of the system on the other hand satisfies equations

$$H|+\rangle = E|+\rangle + i\delta|-\rangle, \;\; H|-\rangle = -i\delta|+\rangle + E|-\rangle.$$

where E ja δ are real numbers.

(a) (2p.) Write the matrix representations of the operators A and H in the basis given by the states $|+\rangle$ and $|-\rangle$.

(b) (2p.) Let the system be in the state $|+\rangle$. If the energy of the system is now measured, what are the possible outcomes of the measurements and their probabilities?

(c) (2p.) Denote the energy eigenvalues as ϵ_+ and ϵ_- ($\epsilon_+ > \epsilon_-$), and the corresponding eigenstates as $|\epsilon_+\rangle$ and $|\epsilon_-\rangle$. If the energy measurement in part (b) resulted in ϵ_+, what is the state vector of the system immediately after the measurement? If A is now measured, what are the possible outcomes and their probabilities?

2. Consider a particle in a quantum mechanical symmetric double well, which has two orthonormal bound states, even and odd ($|E\rangle$ and $|O\rangle$). Let their eigenenergies be E_1 and E_2, i.e.

$$\begin{aligned} H|E\rangle &= E_1|E\rangle, \\ H|O\rangle &= E_2|O\rangle. \end{aligned}$$

(a) (1p.) Suppose that at time $t = 0$ the system is in the state

$$|\Psi(0)\rangle = \alpha|E\rangle + \beta|O\rangle,$$

where $|\alpha|^2 + |\beta|^2 = 1$. What is the state vector $|\Psi(t)\rangle$ at later times $t > 0$?

(b) (3p.) When the system is in the state $(|E\rangle + |O\rangle)/\sqrt{2}$, the probability to observe the particle in the leftmost well is equal to 1. When the state of the system is $(|E\rangle - |O\rangle)/\sqrt{2}$, the probability to find the system from the rightmost well is equal to 1. Suppose that the system is intially, at $t = 0$, prepared so that the particle is certainly in the leftmost well, i.e. $\alpha = \beta = 1/\sqrt{2}$. Compute the probability that at later times, $t > 0$, the particle is found from the rightmost well. How long does it take, that the particle is found from the rightmost well with probability 1?

(c) (2p.) Suppose that the particle was found from the rightmost well at time t. What is the probability to find the particle in the leftmost well at time $3t$, if no measurements are performed on the system during the time interval from t to $3t$?

3. (a) (3p.) Let H be the Hamiltonian of the system and let A represent some observable. Derive the Ehrenfest theorem

$$\frac{d\langle A\rangle}{dt} = -\frac{i}{\hbar}\langle [A,H]\rangle + \langle \frac{\partial A}{\partial t}\rangle.$$

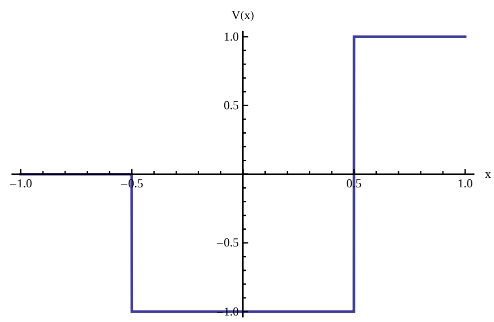
(b) (3p.) Consider the potential

$$V(x) = \begin{cases} 0, & x < -L/2, \\ -V_0, & |x| < L/2, \\ V_0, & x > L/2. \end{cases}$$

The attached figure corresponds to values $V_0 = 1$ eV and $L = 0.5$ nm. The scattering solutions and bound state solutions satisfy the time independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x).$$

Write the most general normalizable solution of the Schrödinger equation when $E < 0$ (but $E > -V_0$). Which conditions the wave function and its derivative have to satisfy at $x = \pm L/2$?



4. The Hamiltonian of the one dimensional harmonic oscillator is

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2.$$

Define the raising and lowering operators as

$$a = \sqrt{\frac{m\omega}{2\hbar}}X + \frac{i}{\sqrt{2\hbar m\omega}}P, \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - \frac{i}{\sqrt{2\hbar m\omega}}P.$$

(a) (1p.) Derive the commutation relation $[a, a^\dagger] = 1$.

(b) (2p.) Consider the number operator $N = a^\dagger a$ and its eigenstates satisfying $N|n\rangle = n|n\rangle$. Show that a lowers and $a^\dagger$ raises the eigenvalue by one, i.e.

$$Na|n\rangle = (n-1)a|n\rangle, \qquad Na^\dagger|n\rangle = (n+1)a^\dagger|n\rangle.$$

(c) (1p.) Determine the constants C and C' so that the states $|n-1\rangle = Ca|n\rangle$ and $|n+1\rangle = C'a^\dagger|n\rangle$ are normalized.

(d) (2p.) Compute the expectation values of the operators X and X^2 in the state $|n\rangle$.

Useful formulas:

$$e^{ix} = \cos x + i \sin x \qquad e^{i\pi} + 1 = 0$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \qquad \cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\frac{1}{2}(1-\cos x) = \sin^2\frac{x}{2} \qquad \frac{1}{2}(1+\cos x) = \cos^2\frac{x}{2}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$\int_{-\infty}^{\infty} dx\, e^{-cx^2} = \sqrt{\frac{\pi}{c}} \qquad \int_{-\infty}^{\infty} dx\, x^2 e^{-cx^2} = \frac{\sqrt{\pi}}{2c^{\frac{3}{2}}}$$

$$\int_{-\infty}^{\infty} dx\, \delta(x-x_0) f(x) = f(x_0) \qquad \int_{-\infty}^{\infty} dx\, \delta(x-x_0) = 1$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{ikx}\tilde{f}(k)$$

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$U(t) = e^{-iHt/\hbar}$$

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \qquad \Delta A \Delta B \geq \frac{1}{2}|\langle[A,B]\rangle|$$

$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\frac{d\langle A\rangle}{dt} = -\frac{i}{\hbar}\langle[A,H]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle$$

$$H = \frac{P^2}{2m} + V(x) \qquad [X,P] = i\hbar$$

$$\psi(x) = \langle x|\psi\rangle \qquad X = x \qquad P = -i\hbar\frac{d}{dx}$$

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$c = 2.998 \cdot 10^8 \text{ m/s} \qquad \hbar = 1.055 \cdot 10^{-34} \text{ Js} \qquad q_e = 1.602 \cdot 10^{-19} \text{ C}$$