Exam: 4 problems, 4 hours

1. Consider a quantum-mechanical two-state system. An observable A of the system has the eigenstates $|+\rangle$ and $|-\rangle$, which satisfy

$$A|+\rangle = |+\rangle, \quad A|-\rangle = -|-\rangle.$$

   (a) (1p.) Write down the matrix representing the operator A in the basis given by the eigenstates $|+\rangle$ and $|-\rangle$. Check that A is Hermitian.

   (b) (3p.) The Hamiltonian of the system is

$$H = \begin{pmatrix} E & -i\delta \\ i\delta & E \end{pmatrix},$$

   where E and δ are real numbers. Let the system be in the state $|+\rangle$. If the energy of the system is measured, what are the possible results, and the probabilities for observing them?

   (c) (2p.) Denote the energy eigenvalues by ϵ_+ and ϵ_- ($\epsilon_+ > \epsilon_-$), and the corresponding eigenstates by $|\epsilon_+\rangle$ and $|\epsilon_-\rangle$. If the energy measurement of part (b) yielded the result ϵ_+, what is the state vector of the system immediately after the measurement? If A is now measured from the system, what results can be obtained, and with what probabilities?

2. Consider an electron, which is described by the Hamiltonian

$$H = \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   To the system are also related the observables

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

   both of which have the eigenvalues $\pm\hbar/2$.

   (a) (2p.) At $t = 0$ let the electron be in the eigenstate of S_x corresponding to the eigenvalue $+\hbar/2$. What is the probability of finding the result $+\hbar/2$, if S_y is measured at time t?

   (b) (1p.) What is the state vector of the system immediately after the measurement performed in part (a), if the result of the measurement was $+\hbar/2$?

   (c) (2p.) Suppose that after the measurement of part (a) (which gave the result $+\hbar/2$) we wait for a time T and measure S_y again. What is the probability of the result being $+\hbar/2$? For what values of T do we get the result $+\hbar/2$ with certainty (that is, with probability 1)?

   (d) (1p.) Is S_y a constant of motion? Explain your answer.

3. (a) (3p.) Starting from the time-dependent Schrödinger equation of wave mechanics, carry out the separation of space and time variables, and show that the wave functions of energy eigenstates have the form

$$\Psi(x,t) = e^{-iEt/\hbar}\psi(x),$$

where $\psi(x)$ satisfies the stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x).$$

(b) (3p.) Show that the probability density $\rho(x,t) = |\Psi(x,t)|^2$ and the probability current

$$j(x,t) = -\frac{i\hbar}{2m}\left(\Psi^*(x,t)\frac{\partial}{\partial x}\Psi(x,t) - \Psi(x,t)\frac{\partial}{\partial x}\Psi^*(x,t)\right)$$

of a wave function $\Psi(x,t)$ satisfy the continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x}.$$

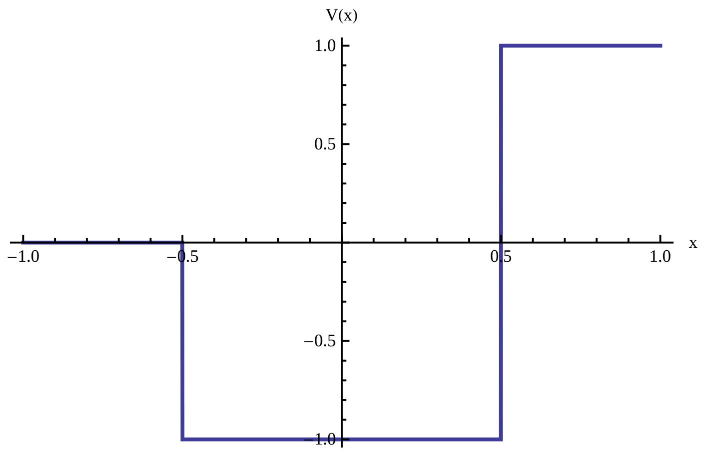
4. (a) (3p.) Consider the potential

$$V(x) = \begin{cases} 0, & x < L/2, \\ -V_0, & |x| < L/2, \\ V_0, & x > L/2. \end{cases}$$

In the figure below we have chosen $V_0 = 1$ eV and $L = 0.5$ nm. The scattering states and bound states satisfy the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x).$$

Write down the most general normalizable solution of the Schrödinger equation, when $E < 0$ (but $E > -V_0$). What conditions must the wave function and its derivative satisfy at the points $x = \pm L/2$?



(b) (3p.) The Hamiltonian of the one-dimensional harmonic oscillator is

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2.$$

Define the lowering and raising operators

$$\begin{aligned} a &= \sqrt{\frac{m\omega}{2\hbar}} X + \frac{i}{\sqrt{2\hbar m\omega}} P, \\ a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} X - \frac{i}{\sqrt{2\hbar m\omega}} P. \end{aligned}$$

Derive the commutation relation $[a, a^\dagger] = 1$. Consider then the eigenstates of the number operator $N = a^\dagger a$, satisfying $N|n\rangle = n|n\rangle$. Show that a lowers and $a^\dagger$ raises the eigenvalue by one; that is,

$$Na|n\rangle = (n-1)a|n\rangle, \qquad Na^\dagger|n\rangle = (n+1)|n\rangle.$$

Useful information:

$$e^{ix} = \cos x + i \sin x \qquad e^{i\pi} + 1 = 0$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \qquad \cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin(2x) = 2 \sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\frac{1}{2}(1 - \cos x) = \sin^2 \frac{x}{2} \qquad \frac{1}{2}(1 + \cos x) = \cos^2 \frac{x}{2}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\int_{-\infty}^{\infty} dx\, e^{-cx^2} = \sqrt{\frac{\pi}{c}} \qquad \int_{-\infty}^{\infty} dx\, x^2 e^{-cx^2} = \frac{\sqrt{\pi}}{2c^{\frac{3}{2}}}$$

$$\int_{-\infty}^{\infty} dx\, \delta(x - x_0) f(x) = f(x_0) \qquad \int_{-\infty}^{\infty} dx\, \delta(x - x_0) = 1$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{ikx} \tilde{f}(k)$$

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

$$U(t) = e^{-iHt/\hbar}$$

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \qquad \Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$$

$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\frac{d\langle A \rangle}{dt} = -\frac{i}{\hbar} \langle [A, H] \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle$$

$$H = \frac{P^2}{2m} + V(x) \qquad [X, P] = i\hbar$$

$$\psi(x) = \langle x | \psi \rangle \qquad X = x \qquad P = -i\hbar \frac{d}{dx}$$

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x) \psi(x,t)$$

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$c = 2.998 \cdot 10^8 \text{ m/s} \qquad \hbar = 1.055 \cdot 10^{-34} \text{ Js} \qquad q_e = 1.602 \cdot 10^{-19} \text{ C}$$