Exam: 4 problems, 4 hours

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1. (a) (2p.) A translation of a system through an infinitesimal distance ϵ is represented by the operator
$$T(\epsilon) = 1 - \frac{i\epsilon}{\hbar}P.$$
Starting from this, show that a translation by a finite distance a is represented by the operator
$$T(a) = e^{-iPa/\hbar}.$$
You may make use of the known identity
$$\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = e^x.$$

   (b) (2p.) Check that in the position representation, $T(a) = e^{-a(d/dx)}$. Then use a series expansion to show that the operation of $T(a)$ on a wave function $\psi(x)$ gives
$$T(a)\psi(x) = \psi(x-a).$$

   (c) (2p.) Suppose that the wave function $\psi(x)$ describes a system placed at x_0. Explain why the wave function $\psi(x-a)$ then represents the system placed at $x_0 + a$ (and not at $x_0 - a$).

2. Consider a system consisting of a spin-1 particle (particle 1) and a spin-1/2 particle (particle 2).

   (a) (2p.) Write all the states in the coupled basis in terms of the uncoupled states. (You may look up the Clebsch–Gordan coefficients from the table.)

   (b) (2p.) Let the Hamiltonian of the system be
$$H = \frac{E_0}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2,$$
where E_0 is a constant, and $\vec{S}_1$ and $\vec{S}_2$ are the spin operators of particles 1 and 2. If the energy of the system is measured, what results can be obtained?

   (c) (2p.) Suppose that the spins of both particles have been measured along the z-axis, and the results $+\hbar$ for particle 1 and $+\hbar/2$ for particle 2 have been found. If the energy of the system is measured immediately after this, what are the probabilities for obtaining each possible value of the energy?

3. (a) (3p.) The wave function of a bound state of a spherically symmetric potential $V(r)$ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi_{n\ell m}}{\partial r}\right)+\frac{1}{2mr^2}L^2\psi_{n\ell m}+V(r)\psi_{n\ell m}=E\psi_{n\ell m},$$

where the operator L^2 depends only on the angles, and $E < 0$ for bound states. Starting from this, make the substitutions $\psi_{n\ell m}(\vec{r}) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$ and $R_{n\ell}(r) = u_{n\ell}(r)/r$ to derive for the radial wave function $u_{n\ell}(r)$ the equation

$$-\frac{\hbar^2}{2m}\frac{d^2u_{n\ell}(r)}{dr^2}+\left(\frac{\hbar^2\ell(\ell+1)}{2mr^2}+V(r)\right)u_{n\ell}(r)=Eu_{n\ell}(r). \tag{1}$$

Remember that the spherical harmonics $Y_{\ell m}(\theta,\phi)$ are eigenfunctions of L^2 with eigenvalues $\hbar^2\ell(\ell+1)$.

(b) (3p.) Assume that $V(r) \to 0$ as $r \to \infty$, and that for small values of r, $V(r)$ becomes infinite less strongly than $1/r^2$. Derive from Eq. (1) the asymptotic form of the function $u_{n\ell}(r)$ for large and small values of r.

4. (a) (3p.) Consider a harmonic oscillator placed on a weak electric field. Then the unperturbed Hamiltonian is $H_0 = P^2/2m + \frac{1}{2}m\omega^2x^2$ and the perturbation is $V = -q|\vec{E}|x$. Using perturbation theory, calculate the first- and second-order corrections to the energy of the ground state $|0\rangle$.

(b) (3p.) Use the variational method to estimate the ground state energy of the one-dimensional potential $V(x) = \lambda x^4$. Choose the function $\psi_\alpha(x) = e^{-\alpha x^2}$ as your trial function.

Useful information:

$$e^{ix} = \cos x + i \sin x \qquad e^{i\pi} + 1 = 0$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \qquad \cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\frac{1}{2}(1-\cos x) = \sin^2\frac{x}{2} \qquad \frac{1}{2}(1+\cos x) = \cos^2\frac{x}{2}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n f(x_0)}{dx^n}(x-x_0)^n \qquad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\int_0^\infty dx\, x^n e^{-cx} = \frac{n!}{c^{n+1}} \qquad \int_{-\infty}^\infty dx\, e^{-cx^2} = \sqrt{\frac{\pi}{c}}$$

$$\int_{-\infty}^\infty dx\, x^2 e^{-cx^2} = \frac{1}{2}\sqrt{\frac{\pi}{c^3}} \qquad \int_{-\infty}^\infty dx\, x^4 e^{-cx^2} = \frac{3}{4}\sqrt{\frac{\pi}{c^5}}$$

$$\int_{-\infty}^\infty dx\, \delta(x-x_0) f(x) = f(x_0) \qquad \int_{-\infty}^\infty dx\, \delta(x-x_0) = 1$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty dx\, e^{-ikx} f(x) \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty dx\, e^{ikx}\tilde{f}(k)$$

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$U(t) = e^{-iHt/\hbar}$$

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \qquad \Delta A \Delta B \geq \frac{1}{2}|\langle [A,B]\rangle|$$

$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\frac{d\langle A\rangle}{dt} = -\frac{i}{\hbar}\langle [A,H]\rangle + \left\langle \frac{\partial A}{\partial t}\right\rangle$$

$$H = \frac{P^2}{2m} + V(x) \qquad [X,P] = i\hbar$$

$$\psi(x) = \langle x|\psi\rangle \qquad X = x \qquad P = -i\hbar\frac{d}{dx}$$

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$c = 2.998\cdot 10^8\ \mathrm{m/s} \qquad \hbar = 1.055\cdot 10^{-34}\ \mathrm{Js} \qquad q_e = 1.602\cdot 10^{-19}\ \mathrm{C}$$

$$\epsilon_0 = 8.854\cdot 10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{Nm}^2} \qquad \mu_0 = 4\pi\cdot 10^{-7}\ \frac{\mathrm{N}}{\mathrm{A}^2} \qquad m_e = 9.109\cdot 10^{-31}\ \mathrm{kg}$$

$$U(t) = e^{-iHt/\hbar} \qquad T(\vec{a}) = e^{-i\vec{P}\cdot\vec{a}/\hbar} \qquad R_{\vec{n}}(\theta) = e^{-i\theta\vec{n}\cdot\vec{J}/\hbar}$$

$$[J_x, J_y] = i\hbar J_z \qquad [J_y, J_z] = i\hbar J_x \qquad [J_z, J_x] = i\hbar J_y$$

$$J^2 = J_x^2 + J_y^2 + J_z^2 \qquad J_\pm = J_x \pm iJ_y$$

$$[J_z, J^2] = 0 \qquad [J_z, J_\pm] = \pm\hbar J_\pm \qquad [J_+, J_-] = 2\hbar J_z$$

$$J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle \qquad J_z|jm\rangle = m\hbar|jm\rangle$$

$$J_\pm|jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1\rangle$$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\sum_k \epsilon_{ijk}\sigma_k \qquad \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$$

$$H = -\vec{\mu}\cdot\vec{B} \qquad \vec{\mu} = \gamma\vec{S} = g\frac{q}{2m}\vec{S} \qquad g_e \simeq 2$$

$$|j_1 j_2, jm\rangle = \sum_{m_1, m_2} \langle j_1 m_1, j_2 m_2|jm\rangle |j_1 m_1\rangle|j_2 m_2\rangle$$

$$E_n^{(1)} = \langle\psi_n^{(0)}|V|\psi_n^{(0)}\rangle$$

$$E_n^{(2)} = \sum_{m\neq q} \frac{|\langle\psi_m^{(0)}|V|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}} \qquad |\psi_n^{(1)}\rangle = \sum_{m\neq q} \frac{\langle\psi_m^{(0)}|V|\psi_n^{(0)}\rangle^2}{E_n^{(0)} - E_m^{(0)}}|\psi_m^{(0)}\rangle$$

Components of the spin operator for a spin-1 particle:

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Harmonic oscillator:

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 = \hbar\omega(a^\dagger a + \tfrac{1}{2})$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}X + \frac{i}{\sqrt{2\hbar m\omega}}P \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - \frac{i}{\sqrt{2\hbar m\omega}}P$$

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad H|n\rangle = \hbar\omega(n+\tfrac{1}{2})|n\rangle$$

Hydrogen atom:

$$a_0 = \frac{\hbar^2}{me^2} = 5.29\times 10^{-11}\ \text{m} \qquad E_0 = \frac{me^4}{2\hbar^2} = 13.6\ \text{eV}$$

$$\begin{aligned}
\psi_{100}(r) &= \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0} \\
\psi_{200}(r) &= \frac{1}{\sqrt{32\pi a_0^3}}\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0} \\
\psi_{210}(r,\theta) &= \frac{1}{\sqrt{32\pi a_0^3}}\frac{r}{a_0}e^{-r/2a_0}\cos\theta \\
\psi_{21\pm 1}(r,\theta,\phi) &= \mp\frac{1}{\sqrt{32\pi a_0^3}}\frac{r}{a_0}e^{-r/2a_0}\sin\theta e^{\pm i\phi}
\end{aligned}$$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	J=1
		M=+1
+1/2	+1/2	1

m_1	m_2	J=1	J=0
		M=0	M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1
		M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2
		+3/2
+1	+1/2	1

m_1	m_2	3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2
		−3/2
−1	−1/2	1

2×1

m_1	m_2	3
		+3
+2	+1	1

m_1	m_2	3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3
		−3
−2	−1	1

1×1

m_1	m_2	2
		+2
+1	+1	1

m_1	m_2	2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2
		−2
−1	−1	1

2×1/2

m_1	m_2	5/2
		+5/2
+2	+1/2	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2
		−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2
		+2
+3/2	+1/2	1

m_1	m_2	2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2
		−2
−3/2	−1/2	1

3/2×1

m_1	m_2	5/2
		+5/2
+3/2	+1	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2
		−5/2
−3/2	−1	1

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

m_1	m_2	7/2
		+7/2
+2	+3/2	1

m_1	m_2	7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2
		−7/2
−2	−3/2	1

3/2×3/2

m_1	m_2	3
		+3
+3/2	+3/2	1

m_1	m_2	3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3
		−3
−3/2	−3/2	1

2×2

m_1	m_2	4
		+4
+2	+2	1

m_1	m_2	4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	4
		−4
−2	−2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.