FYSA242 Statistical physics exam: 4 problems, 4 hours

1. Answer the following questions briefly:
   (a) (1p.) What does Dulong-Petit law state?
   (b) (1p.) Under which conditions gas can be treated as an ideal gas?
   (c) (1p.) What is equipartition of energy?
   (d) (1p.) How does the interaction with the environment differ in canonical and grand canonical ensembles?
   (e) (1p.) Why does the continuum approximation for energy states break down for ideal bosonic gas at low temperatures?
   (f) (1p.) What is Debye frequency?
   (g) (1p.) What is fermi surface?

2. (a) (2p.) At which temperatures the heat capacity of a crystal is given by $C_V = \gamma T + \alpha T^3$? From which two physical phenomena the two terms arise?
   (b) (3p.) What is the qualitative difference between the assumptions of Einstein and Debye theories, and how is this difference reflected in the theoretical predictions for heat capacity C_V?
   (c) (3p.) Derive the partition function for translational motion of single particle $Z_1 = V\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$. (Hint: continuum approximation in **k**-space and $\int_0^\infty x^2 e^{-ax^2}dx = \frac{\sqrt{\pi}}{4}a^{-3/2}$.)
   (d) (3p.) It is known that the pressure of blackbody radiation is $P = \epsilon(T)/3$, where $\epsilon(T) = E/V$ is the energy density of the radiation. Derive the following results for the entropy of the radiation,

$$S = \frac{4}{3}V\frac{\epsilon(T)}{T}, \qquad S = \left(\frac{V}{3}\right)\left(\frac{d\epsilon(T)}{dT}\right),$$

   and using these obtain the Stefan-Boltzmann law $\epsilon \sim T^4$ for the temperature dependence of the radiation.

3. (a) (4p.) Consider a system in heat- and particle bath, and derive the grand canonical probability distribution. In other words, show that the probability of the state whose energy and particle numbers are E_n and N_n, is

$$p(n) = \frac{e^{-\beta(E_n-\mu N_n)}}{\mathcal{Z}}, \qquad \mathcal{Z} = \sum_n e^{-\beta(E_n-\mu N_n)}.$$

   where $\mathcal{Z}$ is the grand canonical partition function.
   (b) (3p.) Consider non-interacting particles in the grand canonical ensemble. Show that the grand canonical partition function can be written with the help of the canonical partition function $Z(N,T,V)$ as

$$\mathcal{Z} = \sum_{N=0}^{\infty} e^{\beta\mu N} Z(N,T,V)$$

   (c) (3p.) Then show that for classical ideal gas the grand potential is $\Phi = -k_BTe^{\beta\mu}Z_1(T,V)$, where $Z_1(T,V)$ is the canonical single particle partition function. Using this result, compute the average number of particles, $\langle N\rangle$, and derive the ideal gas equation of state.

4. Let us model the conduction electrons of metal as ideal fermion gas. After continuum approximations for the density of states you know that the number of electrons whose energies are in the range $[\epsilon, \epsilon + d\epsilon)$, is

$$n(\epsilon - \mu) g(\epsilon) d\epsilon = \frac{4\pi V}{\hbar^3} (2m)^{3/2} \frac{\sqrt{\epsilon} d\epsilon}{e^{\beta(\epsilon-\mu)} + 1}.$$

(a) (3p.) Suppose that the number of conduction electrons is N. Show that

$$\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{3}{8\pi} \frac{N}{V} \right)^{2/3}.$$

(b) (3p.) Let the temperature be $0 < T \ll T_F \equiv \epsilon_F / k_B$. Compute the energy E. You may apply the Sommerfeld expansion.

$$\int_0^\infty d\epsilon n(\epsilon - \mu)\phi(\epsilon) = \int_0^\mu d\epsilon \phi(\epsilon) + \frac{\pi^2}{6} (k_B T)^2 \phi'(\mu) + \cdot,$$

and the fact that at low temperatures

$$\mu(n, T) = \epsilon_F + \delta\mu = \epsilon_F - \frac{\pi^2}{12} \frac{(k_B T)^2}{\epsilon_F}.$$

The result is

$$E = \int_0^\infty d\epsilon \epsilon g(\epsilon) n(\epsilon - \mu) = E_0 + \frac{\pi^2}{6} (kT)^2 g(\epsilon_F),$$

where $E_0 \equiv \int_0^{\epsilon_F} d\epsilon \epsilon g(\epsilon)$.

(c) (4p.) Estimate the number N_{ex} of electrons excited above the fermi energy. Express your result in terms of the ratio T/T_F. You should get

$$N_{\text{ex}} \sim N \frac{T}{T_F}.$$

Useful formulas

~~Mahdollisesti hyödyllisiä tietoja~~

$$k_B = 1.3805 \times 10^{-23}\ \mathrm{JK}^{-1} \qquad k_B \cdot 300\ \mathrm{K} \approx \frac{1}{40}\ \mathrm{eV} \qquad R = 8.3143\ \mathrm{J/mol\,K}$$

$$0\ ^\circ\mathrm{C} = 273.15\ \mathrm{K} \qquad c = 3 \times 10^8\ \mathrm{m/s} \qquad m_e c^2 = 511\ \mathrm{keV}$$

$$1\ \mathrm{atm} = 1.013 \times 10^5\ \mathrm{Nm}^{-2} \qquad g = 9.81\ \mathrm{ms}^{-2} \qquad \hbar = 1.0545 \times 10^{-34}\ \mathrm{Js}$$

$$\mathrm{d}E = đQ + đW \qquad \mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V \qquad \mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N$$

$$S = k_B \ln \Omega \qquad \ln n! \sim n \ln n - n \qquad \binom{n}{m} = \frac{n!}{n!(n-m)!}$$

$$S = -k_B \sum_r p_r \ln p_r \qquad p_r = \frac{1}{Z} e^{-\beta E_r} \qquad Z = \sum_r e^{-\beta E_r} \qquad \beta = 1/k_B T$$

$$G = E - TS + PV \qquad F = E - TS \qquad F = -k_B T \ln Z \qquad E = -\frac{\partial}{\partial \beta} \ln Z$$

$$PV = N k_B T = nRT \qquad \left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)_{\mathrm{cx}} = \frac{\Delta S}{\Delta V} = \frac{L_{12}}{T \Delta V}$$

$$Z_{\mathrm{cl}} = \frac{1}{N!} Z_1^N \qquad Z_1^{\mathrm{tr}} = V \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} \qquad P(v)\mathrm{d}v = \sqrt{\frac{2}{\pi}} \left(\frac{m}{k_B T}\right)^{3/2} v^2 \exp\left[\frac{-mv^2}{2k_B T}\right] \mathrm{d}v$$

$$p_{N,r} = \frac{1}{\mathcal{Z}} e^{\beta(\mu N - E_{N,r})} \qquad \mathcal{Z} = \sum_{N,r} e^{\beta(\mu N - E_{N,r})} \qquad \Omega = -k_B T \ln \mathcal{Z} = E - TS - \mu N = -PV$$

$$\mathcal{Z} = \prod_r \mathcal{Z}_r = \prod_r \sum_{n_r=0}^{\infty} e^{\beta(\mu - \varepsilon_r) n_r} \qquad \langle n_r \rangle = \frac{1}{e^{\beta(\varepsilon_r - \mu)} \pm 1}$$

$$u(\omega, T)\mathrm{d}\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3 \mathrm{d}\omega}{e^{\beta\hbar\omega} - 1} \qquad \mu = \varepsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \dots\right] \qquad k_B T_C \approx 3.3 \times \frac{\hbar^2}{m} \left(\frac{N}{V}\right)^{2/3}$$

$$f(x,y,z) = 0 \quad \text{"}\Rightarrow\text{"} \quad \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1 \quad \text{ja} \quad \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$e^x = 1 + x + \frac{1}{2!} x^2 + \dots \qquad (1+x)^q = 1 + qx + \frac{1}{2!} q(q-1) x^2 + \dots$$

~~H_2O paineessa 1 atm:~~
~~sulamislämpö 335 kJ/kg~~
~~höyrystymislämpö 2260 kJ/kg~~
~~0 °C:ssa veden tiheys 0.9999 g/cm^3 ja jään tiheys 0.9168 g/cm^3~~
~~100 °C:ssa veden tiheys 0.9588 g/cm^3 ja höyryn tiheys 0.0005977 g/cm^3~~