Quantum field theory - I, fall 2013, 2. Exam 13.12.2013

Exam lasts 4 hours. Problems are given only in English, but you can give your answers in Finnish.

Note that a large number of equations and definitions are given in the appendix. All these results can be freely used without derivation.
It is unlikely that you can finish all problems and *renormalization* will be applied to the final score. Hence: leave room only under two conditions: **1) you have produced a complete paper or 2) the bell rings.**

1a. (2p) Explain shortly the following:

- What is content of renormalization procedure from the point of view of the bare and observable parameters of the theory.
- Why is gauge fixing needed and how does that affect gauge field propagator.
- What is a Faddeev-Popov ghost?

1b. (2p) Show by a direct application of LSZ-relations (see appendix) that a counterterm interaction

$$\mathcal{L} = \frac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{2}\delta_M^2\phi^2$$

gives rise to a Feynman rule.

$i\delta_\phi p^2 + i\delta_M^2$

2. (4p) Consider an interaction of the form

$$\mathcal{L}_{\text{int}} = g\phi\bar{\psi}\psi$$

where ϕ is a real scalar and ψ a fermion field. Compute the superficial degree of divergence D for the 3-point function (with one external scalar and two external fermion lines) in this theory. Show that this function is renormalizable precisely when the mass dimension of g is zero. (That is, show that if $d > 0$ the degree of divergence of the generic graph contributing to this function grows as a function of number of internal loops.)

3. (4p) Compute the path integral

$$\int \prod_{i=1}^{n} d\bar{\theta}_i d\theta_i \; e^{\bar{\theta}A\theta - \bar{\eta}\theta - \bar{\theta}\eta}$$

when A is a symmetric $n \times n$-matrix and θ_i, $\bar{\theta}_i$, η_i and $\bar{\eta}_i$ are independent Grassmann-valued variables.

4. (4p) W-boson couples to fermions through a chiral interaction

$$\mathcal{L}_{\rm int} = -\frac{g}{2\sqrt{2}} \bar{f}\gamma^\mu(1-\gamma_5) f W_\mu \,.$$

Compute the decay width of the W-boson (averaged over polarization) into an arbitrary fermion pair $f\bar{f}$ where f has a mass m_f. You will need the massive gauge-boson polarization sum:

$$\sum_\lambda \epsilon_\mu^\lambda(W,k)(\epsilon_\mu^\lambda(W,k))^* = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2}\,.$$

5. (8p) Consider a theory

$$\begin{aligned} \mathcal{L} \;=\; & \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{\lambda_h}{4!}\phi^4 + \frac{1}{2}(\partial_\mu s)^2 - \frac{1}{2}m_s^2 s^2 - \frac{\lambda_s}{4!}s^4 \\ & -\frac{g_\phi}{6!}\phi^6 - \frac{g_s}{6!}s^6 - \frac{g_{\phi s}^{(1)}}{2!4!}\phi^4 s^2 - \frac{g_{\phi s}^{(2)}}{2!4!}s^4\phi^2 + \Delta\mathcal{L}\,. \end{aligned} \tag{1}$$

in $d=3$ dimensions. Justify the form of the counter term Lagrangian:

$$\Delta\mathcal{L} = \frac{1}{2}\delta_\phi[(\partial_\mu\phi)^2 - m_\phi^2\phi^2] - \frac{1}{2}\delta_{m_\phi}^2\phi^2 - \frac{1}{4!}\delta_{\lambda_h}\phi^4 + \ldots$$

where

$$\delta_\phi = Z_\phi - 1; \qquad \delta_{m_\phi} = Z_\phi \delta m_\phi^2; \qquad \delta_{\lambda_h} = \lambda_h(Z_\phi^2 Z_{\lambda_h} - 1)\,.$$

There are of course many more counterterms in this theory, but you need to concentrate only on those indicated above.

- Derive the mass dimensions of the fields ϕ and s in $d = 3-\epsilon$ dimensions.
- Rewrite the Lagrangian using new parameters $\lambda_{i\epsilon} \equiv \mu^\alpha\lambda_i$ and $g_{j\epsilon} \equiv \mu^\beta g_j$, with α and β chosen such that λ_i and g_j keep their dimensions fixed to their $d=3$ values.
- Use the new couplings to define your Feynman rules for the theory.
- Draw diagrammatic expansions for the 2-point function for ϕ-field to first order in couplings λ_i and g_j and for the 4-point function to first order in couplings g_j.
- Compute these diagrams and regularize them using dimensional regularization.
- Compute the counter terms δ_ϕ, δ_{m_ϕ} and δ_{λ_ϕ} from the renormalization conditions for the propagator and the 4-point function at $p=0$:

$$\Delta_\phi^{-1}|_{p^2=0} \equiv -m^2\,, \qquad \frac{\mathrm{d}\Delta_\phi^{-1}}{\mathrm{d}p^2}|_{p^2=0} \equiv 1 \qquad \text{and} \qquad \Gamma_\phi^{(4)}|_{p^2=0} \equiv -i\lambda_\phi\,.$$

Collection of useful (and not so useful) equations

- Free Klein-Gordon theory for real (ϕ) and complex (φ) scalar fields:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2\,, \qquad \mathcal{L} = |\partial_\mu \varphi|^2 - m^2 |\varphi|^2\,.$$

- Free Dirac theory

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \qquad (i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \qquad -i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0$$

- Equation of motion, Hamilton, etc

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \qquad \pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \qquad H = \int d^3x [\pi_\phi \dot{\phi} - \mathcal{L}]$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0 \qquad \pi_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \qquad H = \int d^3x [\pi_\varphi \dot{\varphi} + \pi_\varphi^\dagger \dot{\varphi}^\dagger - \mathcal{L}]$$

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0 \qquad \pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \qquad H = \int d^3x [\pi_\psi \dot{\psi} - \mathcal{L}]$$

$$\Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi}\Delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial_\mu \Delta\psi$$

- Field operators and commutation rules:

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_p}\left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)$$

$$[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}^\dagger] = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \qquad [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}] = [\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{p}'}^\dagger] = 0\,.$$

$$\hat{\varphi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_p}\left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{b}_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)$$

$$[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}^\dagger] = [\hat{b}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}^\dagger] = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$$

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left(\hat{a}_{\mathbf{p}}^s u_{\mathbf{p}}^s e^{-ip\cdot x} + \hat{b}_{\mathbf{p}}^{s\dagger} v_{\mathbf{p}}^s e^{ip\cdot x}\right)$$

$$\{\hat{a}_{\mathbf{p}}^s, \hat{a}_{\mathbf{p}'}^{s'\dagger}\} = \{\hat{b}_{\mathbf{p}}^s, \hat{b}_{\mathbf{p}'}^{s'\dagger}\} = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')\delta_{s,s'}$$

In each of these cases an equal momentum commutator is to be understood as $[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}}^\dagger] = 2E_{\mathbf{p}}V$, where V is the volume of the space. Similar reasoning holds for anticommutators and antiparticle operators.

- Feynman propagators

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

- Contractions

$$\hat{\phi}(x)\hat{\phi}(y) = D_F(x-y) \quad \text{and} \quad \hat{\psi}(x)\hat{\bar{\psi}}(y) = S_F(x-y)$$

- Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

- Weyl representation for gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3.$$

- $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.
- Trace-identities

$$\begin{aligned} Tr[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu} \\ Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \\ Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] &= -4i\epsilon^{\mu\nu\rho\sigma} \end{aligned}$$

- Spinor identities

$$\sum_s u^s_{\mathbf{p}}\overline{u}^s_{\mathbf{p}} = \not{p} + m \qquad \sum_s v^s_{\mathbf{p}}\overline{v}^s_{\mathbf{p}} = \not{p} - m$$

$$\overline{u}^s_{\mathbf{p}}u^{s'}_{\mathbf{p}} = 2m\delta_{ss'} \qquad \overline{v}^s_{\mathbf{p}}v^{s'}_{\mathbf{p}} = -2m\delta_{ss'} \qquad \overline{u}^s_{\mathbf{p}}v^{s'}_{\mathbf{p}} = 0$$

- Mandelstam variables for $12 \to 34$ scattering:

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2.$$

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

- LSZ-reduction formula for real scalar fields

$$\begin{aligned} S_{\rm fi}(l \to m) &\equiv {}_{\rm out}\langle q_1, ..., q_m | p_1, ..., p_l \rangle_{\rm in} \\ &= \prod_{i=1}^{l} \int \! {\rm d}^4 y_j \, e^{iq_j \cdot y_j} \frac{(\Box_j + m_{\rm P}^2)}{-iR} \prod_{j=1}^{m} \int \! {\rm d}^4 x_i \, e^{-ip_i \cdot x_i} \frac{(\Box_i + m_{\rm P}^2)}{-iR} \times \\ &\qquad \times \langle \Omega | \, T(\hat{\phi}(y_1), ..., \hat{\phi}(y_m) \hat{\phi}(x_1), ..., \hat{\phi}(x_l)) \, | \Omega \rangle \end{aligned}$$

- Differential cross section for $2 \to 2$ process

$$\frac{{\rm d}\sigma}{{\rm d}\Omega_{\rm CM}} = \frac{|T|^2}{64\pi^2 s} \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)} S \,, \qquad \frac{{\rm d}\sigma}{{\rm d}t} = \frac{|T|^2}{16\pi \lambda(s, m_1^2, m_2^2)} S \,,$$

where $\lambda(x, y, z) \equiv (x - y - z)^2 - 4yz$ and $S = 1$ for nonidentical and $S = 1/2$ for identical particles in the final state.

- Decay width for $1 \to 2$ process

$$\frac{{\rm d}\Gamma}{{\rm d}\Omega_{\rm CM}} = \frac{|T|^2}{64\pi^2 m_1^3} \lambda^{1/2}(s, m_2^2, m_3^2) S.$$

- d-dimensional solid angle

$$\int {\rm d}^d \Omega = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

- Beta-function representation in terms of Γ

$$\int_0^\infty {\rm d}t \frac{t^{m-1}}{(t + a^2)^\alpha} = \frac{\Gamma(m)\Gamma(\alpha - m)}{\Gamma(\alpha)} (a^2)^{m-\alpha}$$

- Γ-function properties and expansions (rem: $\Gamma(z+1) = z\Gamma(z)$)

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \,,$$

$$\Gamma(\epsilon - 1) = -\frac{1}{\epsilon} + \gamma_E + \mathcal{O}(\epsilon)$$

$$\Gamma(\frac{1}{2} + \epsilon) = \sqrt{\pi} + \mathcal{O}(\epsilon)$$

- Feynmanin parametrization

$$\frac{1}{a_1 a_2 ... a_n} = (n-1)! \int_0^1 \frac{dz_1 dz_2 ... dz_n}{(a_1 z_1 + a_2 z_2 + \cdots + a_n z_n)^n} \delta(1 - \sum_{i=i}^{n} z_i)$$