Quantum Field Theory 1, spring 2011, **1. Exam, 4.3.2011**

Exam lasts 4 hours. Problems are given only in English, but you can of course write your answers in Finnish.
Note that a large number of equations and definitions are given in the appendix. These results can be used freely without a derivation.

1. Give brief answers and explanations to the following:

- *Sketch* (perhaps diagrammatically) the steps that are necessary for establishing a connection between QFT calculable quantities and the physical observables (count rates) in scattering experiments.
- Explain what are the *vacuum* diagrams, how do they arise in the QFT perturabation theory and what is their physical meaning. Give examples in $\lambda\phi^4$-theory.
- What are *connected diagrams* and *1PI-diagrams.* Give examples in $\lambda\phi^4$-theory. What is the role of each type of diagrams in the perturbation expansions for physical processes?

2. Consider free, complex Klein-Gordon scalar field. Prove that the current

$$j^\mu \equiv i\,(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

is conserved, and derive then a representation for the conserved charge

$$\hat{Q} = q \int d^3x : j^0 :$$

in terms of creation and annihilation operators. Here $: j^0 :$ means that j^0 is in normal order.

Let then $|\,\Phi\rangle$ be an eigenstate of the operator $\hat{Q}$ with an eigenvalue Q, *i.e* $\hat{Q}\,|\,\Phi\rangle = Q\,|\,\Phi\rangle$. Show that the creation operator $a^\dagger$ raises the charge of the state $|\,\Phi\rangle$ by q, and that $b^\dagger$ lowers the charge by q. Interpret the operator $\hat{Q}$ physically.

3. a) Compute the dimensions of the fields ϕ, ψ and A_μ (spin-0, -1/2 and 1 fields) when the space-time has a dimension D. (One time and $D-1$ space dimensions, such that in the action $\int \mathrm{d}^4x \to \int \mathrm{d}^Dx$.) Show that the coupling constants related to operators $\bar{\psi}\not{A}\psi$, $\phi(\partial\phi)A$, $\phi^2 A^2$, $A^2(\partial A)$ A^4 are dimensionless only when $D = 4$.

b) Find out what the dimension of the space-time should be so that the theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$$

would be renormalizable. Show that the energy spectrum of the theory is not bounded from below. Hint: compute the Hamiltonian and consider the lowest energy state with a constant ϕ.

4. Consider the Yukawa theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_\phi^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi - g\bar{\psi}\phi\psi\,.$$

The LSZ reduction formula for the decay amplitude $\phi(k) \to \psi(p_1)\bar{\psi}(p_2)$ is

$$\,^{\infty}_{\text{out}}\langle p_1, p_2|k\rangle^{-\infty}_{\text{in}} = \int \mathrm{d}^4x\mathrm{d}^4y_1\mathrm{d}^4y_2\, \left[e^{-ik\cdot x}(\partial_x^2 + m_\phi^2)\right]\left[\bar{u}(p_1)e^{ip_1\cdot y_1}(i\partial\!\!\!/_{y_1} - m_\psi)\right] \times$$
$$\times \langle 0|T(\hat{\psi}(y_1)\hat{\bar{\psi}}(y_2)\hat{\phi}(x))|0\rangle \left[(i\overleftarrow{\partial\!\!\!/}_{y_2} + m_\psi)v(p_2)e^{ip_2\cdot y_2}\right]$$

Write down the perturbation expansion formula for the interacting theory Greens function appearing in the integral, and compute the LSZ-amplitude to the lowest nontrivial order in the perturbation theory in the Yukawa theory. Draw the corresponding Feynman diagram.

5. Consider a theory described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_\phi^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \sum_c \left[\bar{\psi}_c(i\partial\!\!\!/ - m_c)\psi_c - g\bar{\psi}_c\gamma^5\phi\psi_c\right].$$

where ψ_c with $c = a, b, \ldots$ are some fermionic fields.

Part a)
i) Extract the Feynman rules for this theory. *ii)* Draw the Feynman diagram(s) to order g^2 for the annihilation process $a\bar{a} \to b\bar{b}$ in this theory and write down the corresponding T-matrix element(s) using your Feynman rules. *iii)* Compute the square of the spin-averaged matrix element and finally the total spin-averaged cross section $\sigma(s)$ for the process. (Check equation collection for useful formulae.)

Part b)
Draw all diagrams to order g^2 and λg^2 for the scattering $a\bar{a} \to \phi\phi$ and compute the symmetry factor for each diagram.

- Free Klein-Gordon theory (real (ϕ) and complex (φ) scalar fields)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 , \qquad \mathcal{L} = |\partial_\mu \varphi|^2 - m^2 |\varphi|^2 .$$

- Free Dirac theory

$$\mathcal{L}_{\text{Dirac}} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \qquad (i\gamma^\mu \partial_\mu - m)\Psi(x) = 0 \qquad -i\partial_\mu \overline{\Psi}\gamma^\mu - m\overline{\Psi} = 0$$

- Equation of motion, Hamilton, etc

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \qquad \pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \qquad H = \int d^3x [\pi_\phi \dot{\phi} - \mathcal{L}]$$

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0 \qquad \pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \qquad H = \int d^3x [\pi_\psi \dot{\psi} - \mathcal{L}]$$

$$\Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi}\Delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial_\mu \Delta\psi$$

- Real scalar field (ϕ)

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p}\left(a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \qquad [a_{\mathbf{p}}, a_{\mathbf{p}'}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{p}'}^\dagger] = 0 .$$

- Complex scalar field (φ)

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p}\left(a_{\mathbf{p}} e^{-ip\cdot x} + b_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = [b_{\mathbf{p}}, b_{\mathbf{p}'}^\dagger] = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$$

- Dirac field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p}\sum_s \left(a_{\mathbf{p}}^s u_{\mathbf{p}}^s e^{-ip\cdot x} + b_{\mathbf{p}}^{s\dagger} v_{\mathbf{p}}^s e^{ip\cdot x}\right)$$

$$\{a_{\mathbf{p}}^s, a_{\mathbf{p}'}^{s'\dagger}\} = \{b_{\mathbf{p}}^s, b_{\mathbf{p}'}^{s'\dagger}\} = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')\delta_{s,s'}$$

- Feynman propagators

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip\cdot(x-y)}$$

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}e^{-ip\cdot(x-y)}$$

- Contractions

$$\underbrace{\hat{\phi}(x)\hat{\phi}(y)} = D_F(x-y) \quad \text{and} \quad \underbrace{\hat{\psi}(x)\hat{\bar{\psi}}(y)} = S_F(x-y)$$

- Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

- Weyl representation for gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3.$$

- $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

- Trace-identities

$$\begin{aligned} Tr[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu} \\ Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \\ Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] &= -4i\epsilon^{\mu\nu\rho\sigma} \end{aligned}$$

- Spinor identities

$$\sum_s u^s_{\mathbf{p}}\overline{u}^s_{\mathbf{p}} = \not{p} + m \qquad \sum_s v^s_{\mathbf{p}}\overline{v}^s_{\mathbf{p}} = \not{p} - m$$

$$\overline{u}^s_{\mathbf{p}}u^{s'}_{\mathbf{p}} = 2m\delta_{ss'} \qquad \overline{v}^s_{\mathbf{p}}v^{s'}_{\mathbf{p}} = -2m\delta_{ss'} \qquad \overline{u}^s_{\mathbf{p}}v^{s'}_{\mathbf{p}} = 0$$

- Mandelstam variables for $12 \to 34$ scattering:

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2.$$

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

- Differential cross sections :

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{|T|^2}{64\pi^2 s}\frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)}, \qquad \frac{d\sigma}{dt} = \frac{|T|^2}{16\pi\lambda(s, m_1^2, m_2^2)},$$

where $\lambda(x, y, z) \equiv (x - y - z)^2 - 4yz$.