When answering the questions please keep in mind the following:

- Mark your name on each paper you use, but *don't* write down your social security number, even though there is a text box for it in the answer sheet.
- Explain your reasoning carefully. Without the reasoning an answer is worth nothing.
- Check your results whenever possible.

Use of calculators is not allowed in the exam! The available time is four hours.

1. **a)** Let f be an odd function and g an even function. Show that fg is an odd function. *[4p]*

b) Explain in a few sentences what is the Bernoulli–l'Hôpital rule for calculating limits, and under which conditions it can be used. *[4p]*

c) By using the chain rule, derive the formula for the derivative of an inverse function. *[4p]*

2. Let

$$f(x) = \frac{2x-1}{x^2-x}.$$

a) Find the zeros of f. *[2p]*

b) Calculate $f'(x)$ and $f''(x)$. *[4p]*

c) Find the second order Taylor polynomial of f about $x = \frac{1}{2}$. *[6p]*

3. Evaluate the following definite integrals: *[for each, 3p]*

a) $\displaystyle\int_0^1 x \ln x \, dx$ **b)** $\displaystyle\int_{-2}^{2} (t^2 - 2t + 1)^{-1} \, dt$ **c)** $\displaystyle\int_0^{\ln 2} \frac{e^x}{e^{-x}\sqrt{e^{2x}-1}} \, dx$ **d)** $\displaystyle\int_{-\pi}^{\pi} x^3 \cos x \, dx$

4. For a function $f : \mathbb{R} \to \mathbb{R}$ the following properties are known:

i) It is an even function.

ii) It is continuous but not differentiable at $x = 0$.

iii) Its smallest value (the absolute minimum value) is 1.

iv) $f'(x) = (1-x)e^{-x}$ for each $x > 0$.

Determine what is the function in question and check that it has properties *i)–iv)*. *[12p]*

★ **Special question.** Evaluate the definite integral

$$\int_0^1 x^x \, dx$$

as accurately as you can. *[+4p]*

From this question you can earn 4 extra points. These points are not calculated in the total maximum score of the exam which is 4 · 12 = 48 points.

Some useful formulas on the other side

Elementary functions

$$n! = n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 2\cdot 1$$
$$a^b = \exp(b\ln a) \quad (a>0)$$

$$\sin^2 x + \cos^2 x = 1$$
$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = 1 - 2\sin^2 x$$
$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$
$$\sin(x\pm y) = \sin x\cos y \pm \cos x\sin y$$
$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right)$$
$$\cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right)$$
$$\cosh^2 x - \sinh^2 x = 1$$

Partial fraction decomposition

Let $P(x)/Q(x)$ be a rational function with $\deg P < \deg Q$.

– If $Q(x) = (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_m)$, where all zeros are of multiplicity 1, then there exists numbers $c_1,\dots,c_m$ such that

$$\frac{P(x)}{Q(x)} = \frac{c_1}{x-\lambda_1} + \frac{c_2}{x-\lambda_2} + \dots + \frac{c_m}{x-\lambda_m}.$$

– If Q has a zero λ of multiplicity p, the term in the decomposition corresponding to λ is of the form

$$\frac{d_1}{x-\lambda} + \frac{d_2}{(x-\lambda)^2} + \dots + \frac{d_p}{(x-\lambda)^p}.$$

– If the factorization of Q has quadratic factors $R_1,\dots,R_k$ with no real zeros, add terms

$$\frac{a_1x+b_1}{R_1(x)} + \frac{a_2x+b_2}{R_2(x)} + \dots + \frac{a_kx+b_k}{R_k(x)}.$$

Special functions

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

Differentiation

$f(x)$	$f'(x)$	
x^a	ax^{a-1}	$(a\neq 0)$
e^x	e^x	
$\ln x$	$\dfrac{1}{x}$	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\dfrac{1}{\cos^2 x}$	
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$	
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$	
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\tanh x$	$\dfrac{1}{\cosh^2 x}$	
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$	
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$	
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$	

Integration

See *Differentiation*.

Applications of differentiation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$f(x) = P_n(x) + \frac{1}{(n+1)!}f^{(n+1)}(s)(x-a)^{n+1}$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \quad (x\in\,]-1,1])$$
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad (|x|<1)$$