1. intermediate exam (1. välikoe): 4 problems, 4 hours

1. As you remember, the Hamilton operator for a charged particle interacting with an electromagentic field can be obtained through the minimal substitution principle,

$$\hat{H}_0(\hat{\mathbf{p}}, \hat{\mathbf{x}}) \rightarrow \hat{H}_0(\hat{\mathbf{p}} - q\mathbf{A}(\mathbf{x}, t), \hat{\mathbf{x}}) + q\varphi(\mathbf{x}, t),$$

where $\mathbf{A}(\mathbf{x}, t)$ is the vector potential and $\varphi(\mathbf{x}, t)$ is the scalar potential, and q is the charge of the particle. Recall that $\hat{\mathbf{p}} = -i\hbar\nabla$.

a) Applying the minimal substitution principle to the Hamilton operator of a free spin-$\frac{1}{2}$ particle, $\hat{\mathbf{H}} = \frac{1}{2m}(\vec{\sigma} \cdot \hat{\mathbf{p}})^2$, show that the system's Hamilton operator becomes

$$\hat{\mathbf{H}} = \frac{\mathbf{I}_2}{2m}(\hat{\mathbf{p}} - q\mathbf{A}(\mathbf{x}, t))^2 - \frac{q\hbar}{2m}\vec{\sigma} \cdot \mathbf{B}(\mathbf{x}, t) + q\varphi(\mathbf{x}, t)\mathbf{I}_2.$$

Above, $\mathbf{I}_2$ is the 2×2 unit matrix and $\mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t)$ is the magnetic field. Consult the collection of formulae in the end of the question paper for the needed Pauli spin-matrix identity.

b) Show that a gauge transformation

$$\begin{aligned} \tilde{\mathbf{A}}(\mathbf{x}, t) &= \mathbf{A}(\mathbf{x}, t) + \nabla f(\mathbf{x}, t) \\ \tilde{\varphi}(\mathbf{x}, t) &= \varphi(\mathbf{x}, t) - \frac{\partial f(\mathbf{x}, t))}{\partial t} \\ \tilde{\psi}(\mathbf{x}, t) &= e^{i\frac{q}{\hbar}f(\mathbf{x}, t)}\psi(\mathbf{x}, t) \end{aligned}$$

does not change the dynamics of the system, i.e. that it leaves the Schrödinger equation invariant,

$$\hat{\mathbf{H}}(\tilde{\mathbf{A}}, \tilde{\varphi})\tilde{\psi} = i\hbar\frac{\partial\tilde{\psi}}{\partial t}$$

2.a) The asymptotic ($r \rightarrow \infty$) solution of the integral equation for potential scattering is known to be

$$\Psi_{\mathbf{k_i}}(\mathbf{r}) = \Phi_{\mathbf{k_i}}(\mathbf{r}) - \frac{e^{\pm ikr}}{r}\frac{1}{4\pi}\int d^3r' e^{\mp i\mathbf{k_f} \cdot \mathbf{r}'} U(\mathbf{r}')\Psi_{\mathbf{k_i}}(\mathbf{r}'),$$

where $U(\mathbf{r}) = \frac{2\mu}{\hbar^2}V(\mathbf{r})$, $\mathbf{k_i} = k\hat{\mathbf{e}}_z$, $\mathbf{k_f} = k\hat{\mathbf{e}}_\mathbf{r}$, $k^2 = 2\mu E/\hbar^2$, and $\Phi_{\mathbf{k_i}}(\mathbf{r}) = (2\pi)^{-3/2}e^{i\mathbf{k_i}\cdot\mathbf{r}}$. Identify the scattering amplitude $f_k(\theta, \varphi)$ in the above expression and explain in one sentence why we should choose the upper signs in the exponents. Then derive the Born approximation for the scattering amplitude $f_B(\theta, \phi)$.

b) Using the result which you obtained above, compute the scattering amplitude $f_B(\theta, \phi)$ and the differential cross-section $d\sigma/d\Omega$ in the Born approximation for a radially symmetric Yukawa potential

$$V(r) = V_0\frac{e^{-\kappa r}}{r},$$

where V_0 and $\kappa > 0$ are constants.

c) Show explicitly how your result for $d\sigma/d\Omega$ depends on the scattering angle θ and on the energy of the collision. Sketch $d\sigma/d\Omega$ as a function of θ, both at the small-energy limit $E \ll \frac{\hbar^2\kappa^2}{2\mu}$ and at the high-energy limit $E \gg \frac{\hbar^2\kappa^2}{2\mu}$. According to your result, what happens to the backward scattering cross section at high energies?

3. Let's consider the scattering off a delta-function shell potential

$$V(r) = \alpha\delta(r-a),$$

where a and α are constants, in terms of the partial wave analysis. The scattering amplitude and the partial wave amplitudes are known to be

$$f_k(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta), \qquad f_l(k) = e^{i\delta_l(k)}\sin\delta_l(k).$$

Note also that the delta function above applies only to the radial disctance r but not to the angles θ, φ.

a) Starting from the stationary radial Schrödinger equation (see the collection of formulae), compute the s-wave phase shift $\delta_0(k)$. Express your final result for $\tan\delta_0(k)$ in terms of basic trigonometric functions and the dimensionless variable $\beta \equiv \frac{2m\alpha a}{\hbar^2}$. You can use (without deriving it) the fact that the discontinuity of the 1st derivatives of $R(r)$ at $r = a$ is given by $R'(a+\epsilon) - R'(a-\epsilon) = \frac{\beta}{a}R(a)$, where $\epsilon \to 0_+$.

b) Compute the s-wave contribution to the total cross section in the low-energy limit, $ka \ll 1$. Express your result in terms of the constants β and a.

c) Using the exact result which you obtained in the item (a) above, explain briefly when resonant scattering in the s-wave takes place, and write down an equation from which we could (through numerical solution) obtain the value of the energy at which such a resonant scattering happens.

4. Let's put a spinless Hydrogen atom into a weak time-dependent external electric field, which points into the y direction and vanishes asymptotically in time. Let's suppose this gives rise to a perturbation potential

$$\hat{V}_S(t) = C\frac{\hat{y}}{(t^2+\tau_1^2)(t^2+\tau_2^2)},$$

where C and $\tau_2 > \tau_1 > 0$ are real constants. Note that $\hat{y}$ above is the y-coordinate operator (and not a unit vector).

a) Using lowest-order time-dependent perturbation theory, find the selection rules for n, l and m in transitions from the ground state $|1,0,0\rangle$ to any of the excited states. The collection of formulae is most likely again useful.

b) Calculate the probability of a transition from the ground state to a 2p-state $|2,1,1\rangle$ during an infinitely long period of time (set $t_0 \to -\infty$ and $t \to \infty$). In doing the residue integrals, you should explain in sufficient detail why you choose the particular half-plane for closing the integration path. Express the final result in terms of the constants C, τ_1, and τ_2, and the Bohr radius and Hydrogen eigenenergies.

Useful(?) formulas and equations, for any of the problems:

Spherical coordinates and spherical harmonics:

$$\mathbf{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \qquad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{1}{\hbar^2 r^2}\hat{L}^2$$

$$\int d^3r = \int_0^\infty drr^2 \int_{4\pi} d\Omega = \int_0^\infty drr^2 \int_0^\pi d\theta\sin\theta \int_0^{2\pi} d\varphi = \int_0^\infty drr^2 \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\varphi$$

$$\hat{L}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\theta,\varphi) \qquad \hat{L}_z Y_{lm}(\theta,\varphi) = \hbar m Y_{lm}(\theta,\varphi)$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad \int d\Omega\, Y^*_{l'm'}(\theta,\varphi)Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

$$Y_{lm}(\theta,\varphi) = (-1)^{\frac{m+|m|}{2}}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)e^{im\varphi} \qquad Y_{l,-m}(\theta,\varphi) = (-1)^m Y^*_{l,m}(\theta,\varphi)$$

$$P_l^k(z) = (1-z^2)^{k/2}\frac{d^k}{dz^k}P_l(z) \qquad P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l$$

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$$

$$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \qquad Y_{2\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\varphi} \qquad Y_{2\pm2}(\theta,\varphi)\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$$

Stationary Schrödinger equation, the radial part:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1) - r^2\frac{2m}{\hbar^2}V(r)\right]R(r) = 0, \qquad k^2 = \frac{2mE}{\hbar^2}$$

Spherical Bessel & Neumann functions:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1)\right]R(r) = 0 \rightarrow \qquad R(r) = Aj_l(kr) + Bn_l(kr)$$

$$j_l(x) = 2^l x^l \sum_{s=0}^\infty \frac{(-1)^s(s+l)!}{s!(2s+2l+1)!}x^{2s} \qquad n_l(x) = \frac{(-1)^{l+1}}{2^l x^{l+1}}\sum_{s=0}^\infty \frac{(-1)^s(s-l)!}{s!(2s-2l)!}x^{2s}$$

$$j_0(x) = \frac{\sin x}{x} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_0(x) = -\frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

Trigonometric functions:

$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos^2 x + \sin^2 x = 1, \qquad \sin 2x = 2\sin x\cos x$$

Euler: $e^{i\alpha} = \cos\alpha + i\sin\alpha$ $\qquad \cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$ $\qquad \sin\alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$

$$\sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!}$$

See next page!

Hydrogen-like atom wave-functions:

$$\Psi_{nlm_l}(\mathbf{x}) = R_{nl}(r)Y_{lm_l}(\theta,\varphi) \qquad \kappa = \frac{Z}{na} \qquad a = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2} = \frac{\hbar}{\alpha\mu c}$$

$$R_{nl}(r) = \sqrt{(2\kappa)^3\frac{(n-l-1)!}{2n(n+l)!}}(2\kappa r)^l e^{-\kappa r}L^{2l+1}_{n-l-1}(2\kappa r) \qquad L^q_p(x) = \sum_{k=0}^{p}(-1^k)\frac{(p+q)!x^k}{(p-k)!(q+k)!k!}$$

$$R_{10} = 2\left(\frac{Z}{a}\right)^{3/2}e^{-Zr/a} \qquad R_{20} = \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2}\left(1-\frac{Zr}{2a}\right)e^{-Zr/2a} \qquad R_{21} = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2}re^{-Zr/2a}$$

$$\text{Spherical spinors :} \qquad (\mathcal{Y}_{ljm}(\Omega))_{m_s} = \langle\Omega, m_s|l, s=\frac{1}{2}, j, m\rangle_c$$

$$\begin{aligned} |l,s=\frac{1}{2}, j = l\pm\frac{1}{2}, m\rangle_c &= \pm\sqrt{\frac{l\pm m+\frac{1}{2}}{2l+1}}|l,s=\frac{1}{2},m_l=m-\frac{1}{2},m_s=\frac{1}{2}\rangle_u \\ &+ \sqrt{\frac{l\mp m+\frac{1}{2}}{2l+1}}|l,s=\frac{1}{2},m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\rangle_u \end{aligned}$$

$$\int d\Omega\mathcal{Y}_{ljm}(\Omega)^\dagger\mathcal{Y}_{l'j'm'}(\Omega) = \delta_{ll'}\delta_{jj'}\delta_{mm'}$$

For integrations in the complex plane:

$$\mathrm{Res}f(z)\Big|_{z=z_0} = \lim_{z\to z_0}\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}[(z-z_0)^n f(z)] \qquad \oint_C dz f(z) = 2\pi i\sum_{j=1}^{n}\mathrm{Res}f(z)\Big|_{z=z_j}.$$

For integrations

$$\int_0^\infty dx x^n e^{-x} = n! \qquad \int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

Transition probability

$$P_{fi}(t,t_0) = \frac{1}{\hbar^2}\left|\int_{t_0}^{t}dt_1\langle\phi_f|\hat{V}_S(t_1)|\phi_i\rangle e^{i(E_f-E_i)t_1/\hbar}\right|^2 + \mathcal{O}(V_S^2)$$

Generalized angular momentum

$$[\hat{J}_i,\hat{J}_j] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}\hat{J}_k, \qquad [\hat{\mathbf{J}}^2,\hat{J}_i] = 0, \qquad \hat{\mathbf{J}}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \qquad \hat{J}_z|j,m\rangle = \hbar m|j,m\rangle$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \qquad \hat{J}_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$$

Hyperbolic functions

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

Pauli spin-matrix identities

$$(\vec{\sigma}\cdot\ \mathbf{A})(\vec{\sigma}\cdot\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B})\mathbf{I}_2 + i(\mathbf{A}\times\mathbf{B})\cdot\vec{\sigma}$$

$$\sigma_j\sigma_k = \delta_{jk}\mathbf{I}_2 + i\epsilon_{jkl}\sigma_l \qquad [\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l \qquad \vec{\sigma}\times\vec{\sigma} = 2i\vec{\sigma}$$

Vector and Levi-Civita identities

$$(\mathbf{a}\times\mathbf{b})_i = \epsilon_{ijk}a_j b_k \qquad \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$