Final exam (tentti): 5 problems, 4 hours

Regarding all problems:

Remember the *collection of formulae* in the end of the problem sheet.

1. (a) The asymptotic ($r \to \infty$) solution of the integral equation for potential scattering is known to be

$$\Psi_{\mathbf{k_i}}(\mathbf{r}) = \Phi_{\mathbf{k_i}}(\mathbf{r}) - \frac{e^{\pm ikr}}{r}\frac{1}{4\pi}\int d^3r' e^{\mp i\mathbf{k_f}\cdot\mathbf{r}'}U(\mathbf{r}')\Psi_{\mathbf{k_i}}(\mathbf{r}'),$$

where $U(\mathbf{r}) = \frac{2\mu}{\hbar^2}V(\mathbf{r})$, $\mathbf{k_i} = k\hat{\mathbf{e}}_z$, $\mathbf{k_f} = k\hat{\mathbf{e}}_\mathbf{r}$, $k^2 = 2\mu E/\hbar^2$, and $\Phi_{\mathbf{k_i}}(\mathbf{r}) = (2\pi)^{-3/2}e^{i\mathbf{k_i}\cdot\mathbf{r}}$. Identify the scattering amplitude $f_k(\theta,\varphi)$ in the above expression and explain in one sentence why we should choose the upper signs in the exponents. Then derive the Born approximation for the scattering amplitude $f_B(\theta,\phi)$.

(b) Using the result which you obtained above, compute the scattering amplitude $f_B(\theta,\phi)$ and the differential cross-section $d\sigma/d\Omega$ in the Born approximation for a radially symmetric delta-function potential

$$V(r) = \alpha\delta(r-a),$$

where a and α are constants. Note that this delta-function above applies only to the radial distance r but not to the angles θ,φ. Express your final results in terms of the dimensionless constant $\beta \equiv \frac{2\mu\alpha a}{\hbar^2}$, and show the energy and scattering angle dependencies of your result explicitly.

(c) Sketch the behaviour of $d\sigma/d\Omega$ as a function of the scattering angle in the case $ka = 2\pi$, as well as in the high-energy and low-energy limits.

2. A spinless hydrogen atom, which is in the 2p state $|n{=}2, l{=}1, m{=}0\rangle$, is put into a time-dependent perturbing potential

$$\hat{V}_S(t) = C\frac{\hat{z}^2}{t^2+\tau^2},$$

where C and $\tau > 0$ are constants and $\hat{z}$ is the z-coordinate operator.

Using the Gaunt's formula and the attached table of Clebsch-Gordan coefficients, calculate the probability of transition from the state $|2\,1\,0\rangle$ to the highest-l $n{=}4$ state $|4\,l\,0\rangle$ which is allowed according to the lowest-order time-dependent perturbation theory during an infinitely long period of time (set $t_0 \to -\infty$ and $t \to \infty$).

3. (a) Using the Wigner-Eckart theorem, show that for a vector operator $\hat{\mathbf{V}}$, we have

$$\langle \xi jm|\hat{\mathbf{V}}|\xi jm'\rangle = \frac{\langle \xi jm|\hat{\mathbf{V}}\cdot\hat{\mathbf{J}}|\xi jm\rangle}{\hbar^2 j(j+1)}\langle \xi jm|\hat{\mathbf{J}}|\xi jm'\rangle$$

(b) Apply the above result in the following nonrelativistic nuclear shell-model problem, where the system(=nucleus) consists of protons and neutrons (i.e. spin-$\frac{1}{2}$ particles) and their mutual interactions. For those nuclei whose number of protons+neutrons is odd, the oribital angular momentum $\mathbf{L}$ and spin angular momentum $\mathbf{S}$ can be assumed to be those of the last (highest-energy), odd, proton or neutron. Then $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and thus $j = l \pm \frac{1}{2}$, and the quantum numbers in ξ above include l and $s = \frac{1}{2}$. The magnetic moment operator $\hat{\mathbf{M}}$ of the nucleus is defined as

$$\hat{\mathbf{M}} = \gamma_L \hat{\mathbf{L}}/\hbar + \gamma_S \hat{\mathbf{S}}/\hbar,$$

where γ_L and γ_S are constants. The magnetic moment μ of the nucleus is defined as the largest possible absolute value (norm) of the vector $\langle \mathbf{M}\rangle \equiv \langle \xi jm|\hat{\mathbf{M}}|\xi jm\rangle$. Show that for $l = j - \frac{1}{2}$ the magnetic moment μ becomes

$$\mu = (j - \frac{1}{2})\gamma_L + \frac{1}{2}\gamma_S.$$

4. Let's consider the Fock space formulation of the angular momentum operator in a system of identical fermions.

   (a) Angular momentum is an additive quantity, so that the generic form of the Fock space 1-particle operator (see the collection of formulae) holds. Using this, show (in sufficient detail) that the total angular momentum operator can be written as

$$\hat{\vec{\mathcal{J}}} = \sum_{\alpha} \sum_{j} \sum_{m,m'} a^{\dagger}_{\alpha j m'} a_{\alpha j m} \langle \alpha j m' | \hat{\vec{j}} | \alpha j m \rangle,$$

   where $\hat{\vec{j}} = (\hat{j}_x, \hat{j}_y, \hat{j}_z)$ is the angular momentum operator for a 1-particle state, and α stands for all remaining quantum numbers needed to spesify the basis.

   (b) Let's then consider the following 2-particle state in the Fock space:

$$|F^{(2)}\rangle = C \sum_{m_1, m_2} \langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle a^{\dagger}_{\alpha_2 j_2 m_2} a^{\dagger}_{\alpha_1 j_1 m_1} |0\rangle$$

   where $\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle$ is a Clebsch-Gordan coefficient and C is a normalization constant. Show with an explicit calculation that $|F^{(2)}\rangle$ is an eigenstate of $\hat{\mathcal{J}}_z$ with an eigenvalue $\hbar M$. Let's assume here for simplicity that $j_1 \neq j_2$.

5. Starting from the Lorentz-covariant form of the Dirac equation (DE) for a spin-$\frac{1}{2}$ particle in classical electromagnetic field,

$$[\gamma^{\mu}(i\hbar\partial_{\mu} - qA_{\mu}(x)) - mc]\Psi(x) = 0,$$

   show that for the stationary case with time-independent weak electromagnetic field the nonrelativistic (NR) limit of this equation is the Pauli equation,

$$\left[\frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A}(\mathbf{x}))^2 \mathbf{1}_2 - \frac{q\hbar}{2m}\vec{\sigma} \cdot \mathbf{B}(\mathbf{x}) + q\varphi(\mathbf{x})\mathbf{1}_2\right] \psi_{NR}(\mathbf{x}) = E_{NR}\psi_{NR}(\mathbf{x}).$$

   Hints: First bring the DE into the form $i\hbar\partial_0\Psi = ...$, then use the ansatz

$$\Psi(x) = e^{-\frac{i}{\hbar}Et} \begin{pmatrix} \psi_u(\mathbf{x}) \\ \psi_l(\mathbf{x}) \end{pmatrix}$$

   and the Dirac-Pauli representation. Recall also that $A^{\mu} = (\frac{\varphi}{c}, \mathbf{A})$ and $\hat{\mathbf{p}} = -i\hbar\nabla$.

Collection of formulae:

Spherical coordinates and spherical harmonics:

$$\mathbf{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \qquad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{1}{\hbar^2 r^2}\hat{L}^2$$

$$\int d^3r = \int_0^\infty drr^2 \int_{4\pi} d\Omega = \int_0^\infty drr^2 \int_0^\pi d\theta\sin\theta \int_0^{2\pi} d\varphi = \int_0^\infty drr^2 \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\varphi$$

$$\hat{L}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\theta,\varphi) \qquad \hat{L}_z Y_{lm}(\theta,\varphi) = \hbar m Y_{lm}(\theta,\varphi)$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad \int d\Omega\, Y^*_{l'm'}(\theta,\varphi)Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

$$Y_{lm}(\theta,\varphi) = (-1)^{\frac{m+|m|}{2}}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)e^{im\varphi} \qquad Y_{l,-m}(\theta,\varphi) = (-1)^m Y^*_{l,m}(\theta,\varphi)$$

$$P_l^k(z) = (1-z^2)^{k/2}\frac{d^k}{dz^k}P_l(z) \qquad P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l$$

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$$

$$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \qquad Y_{2\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\varphi} \qquad Y_{2\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$$

Stationary Schrödinger equation, the radial part:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1) - r^2\frac{2m}{\hbar^2}V(r)\right]R(r) = 0, \qquad k^2 = \frac{2mE}{\hbar^2}$$

Spherical Bessel & Neumann functions:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1)\right]R(r) = 0 \rightarrow \quad R(r) = Aj_l(kr) + Bn_l(kr)$$

$$j_l(x) = 2^l x^l \sum_{s=0}^\infty \frac{(-1)^s(s+l)!}{s!(2s+2l+1)!}x^{2s} \qquad n_l(x) = \frac{(-1)^{l+1}}{2^l x^{l+1}}\sum_{s=0}^\infty \frac{(-1)^s(s-l)!}{s!(2s-2l)!}x^{2s}$$

$$j_0(x) = \frac{\sin x}{x} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_0(x) = -\frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

Transition probability, lowest order, $i \neq f$:

$$P_{fi}(t,t_0) \equiv |\langle\phi_f|\psi(t)\rangle|^2 \approx \frac{1}{\hbar^2}\left|\int_{t_0}^t dt_1\langle\phi_f|\hat{V}_S(t_1)|\phi_i\rangle e^{i(E_f-E_i)t_1/\hbar}\right|^2$$

Power series, Taylor expansions:

$$e^x = \sum_{n=0}^\infty \frac{x^n}{n!} \qquad \cos x = \sum_{n=0}^\infty (-1)^n\frac{x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^\infty (-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots \qquad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

For integrations:

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}, \qquad \int_{-\infty}^\infty dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$\mathrm{Res} f(z)\big|_{z=z_0} = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}[(z-z_0)^n f(z)] \qquad \oint_C dz f(z) = 2\pi i \sum_{j=1}^n \mathrm{Res} f(z)\big|_{z=z_j}$$

Hydrogen-like atom wave-functions:

$$\Psi_{nlm}(\mathbf{x}) = R_{nl}(r) Y_{lm}(\theta,\varphi) \qquad \kappa = \frac{Z}{na} \qquad a = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

$$R_{nl}(r) = \sqrt{(2\kappa)^3 \frac{(n-l-1)!}{2n(n+l)!}}(2\kappa r)^l e^{-\kappa r} L_{n-l-1}^{2l+1}(2\kappa r) \qquad L_p^q(x) = \sum_{k=0}^p (-1^k)\frac{(p+q)!x^k}{(p-k)!(q+k)!k!}$$

$$R_{10} = 2\left(\frac{Z}{a}\right)^{3/2} e^{-Zr/a} \qquad R_{20} = \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2}\left(1-\frac{Zr}{2a}\right)e^{-Zr/2a} \qquad R_{21} = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r e^{-Zr/2a}$$

$$R_{30} = \frac{2}{3\sqrt{3}}\left(\frac{Z}{a}\right)^{3/2}\left(1-\frac{2Zr}{3a}+\frac{2}{27}(\frac{Zr}{3a})^2\right)e^{-Zr/3a} \qquad R_{31} = \frac{8}{27\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r\left(1-\frac{Zr}{6a}\right)e^{-Zr/3a}$$

$$R_{32} = \frac{4}{81\sqrt{30}}\left(\frac{Z}{a}\right)^{7/2} r^2 e^{-Zr/3a} \qquad R_{40} = \frac{1}{4}\left(\frac{Z}{a}\right)^{3/2}\left(1-\frac{3Zr}{4a}+\frac{1}{8}(\frac{Zr}{a})^2-\frac{1}{192}(\frac{Zr}{a})^3\right)e^{-Zr/4a}$$

$$R_{41} = \frac{\sqrt{5}}{16\sqrt{3}}\left(\frac{Z}{a}\right)^{5/2} r\left(1-\frac{Zr}{4a}+\frac{1}{80}(\frac{Zr}{a})^2\right)e^{-Zr/4a}$$

$$R_{42} = \frac{1}{64\sqrt{5}}\left(\frac{Z}{a}\right)^{7/2} r^2\left(1-\frac{Zr}{12a}\right)e^{-Zr/4a} \qquad R_{43} = \frac{1}{768\sqrt{35}}\left(\frac{Z}{a}\right)^{9/2} r^3 e^{-Zr/4a}$$

Spherical spinors:

$$(\mathcal{Y}_{ljm}(\Omega))_{m_s} = \langle\Omega, m_s|l, s=\frac{1}{2}, j, m\rangle_c \qquad \int d\Omega \mathcal{Y}_{ljm}(\Omega)^\dagger \mathcal{Y}_{l'j'm'}(\Omega) = \delta_{ll'}\delta_{jj'}\delta_{mm'}$$

Trigonometric functions:

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \cos^2 x + \sin^2 x = 1 \quad \sin 2x = 2\sin x\cos x$$

Euler: $e^{i\alpha} = \cos\alpha + i\sin\alpha \qquad \cos\alpha = \frac{1}{2}(e^{i\alpha}+e^{-i\alpha}) \qquad \sin\alpha = \frac{1}{2i}(e^{i\alpha}-e^{-i\alpha})$

Angular momentum:

$$\hat{\mathbf{J}}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \qquad \hat{J}_z|j,m\rangle = \hbar m|j,m\rangle$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \qquad \hat{J}_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$$

$$[\hat{J}_i, \hat{J}_j] = i\hbar\sum_{k=1}^3 \epsilon_{ijk}\hat{J}_k, \quad [\hat{\mathbf{J}}^2, \hat{J}_i] = 0$$

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}_2 \qquad (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\mathbf{1}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

Gaunt's formula:

$$\int d\Omega\, Y^*_{lm}(\Omega) Y_{l_1 m_1}(\Omega) Y_{l_2 m_2}(\Omega) = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\, {}_u\langle l_1 l_2 m_1 m_2 | l_1 l_2 l m\rangle_c\, {}_u\langle l_1 l_2 00 | l_1 l_2 l 0\rangle_c$$

Wignert-Eckart theorem:

$$\langle \xi' j' m' | \hat{T}^{(k)}_q | \xi j m\rangle = \frac{1}{\sqrt{2j'+1}}\, {}_u\langle j k m q | j k j' m'\rangle_c\, \langle \xi' j' || T^{(k)} || \xi j\rangle$$

where

$$\langle \xi' j' || T^{(k)} || \xi j\rangle \equiv \frac{1}{\sqrt{2j'+1}} \sum_{m_1, m_2, q'} \langle \xi' j' m_1 | \hat{T}^{(k)}_{q'} | \xi j m_2\rangle \langle j k m_2 q' | j k j' m_1\rangle$$

$SU(2)$ tensor operator:

$$[\hat{J}_z, \hat{T}^{(k)}_q] = q \hat{T}^{(k)}_q \qquad [\hat{J}_\pm, \hat{T}^{(k)}_q] = \sqrt{k(k+1) - q(q\pm 1)}\, \hat{T}^{(k)}_{q\pm 1},$$

where q refers to the spherical components, which for a vector operator are

$$\hat{V}_{+1} = -\frac{1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y), \qquad \hat{V}_0 = \hat{V}_z \qquad \hat{V}_{-1} = +\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y)$$

Spherical unit vectors:

$$\hat{e}_{\pm 1} = \mp \frac{1}{\sqrt{2}}(\hat{e}_x \pm i \hat{e}_y), \qquad \hat{e}_0 = \hat{e}_z$$

Scalar products in spherical basis: $\mathbf{A}\cdot\mathbf{B} = -A_{+1}B_{-1} - A_{-1}B_{+1} + A_0 B_0$

Fermionic operators in the Fock space:

$$a_\nu | n_1 n_2 \ldots 1_\nu \ldots\rangle = (-1)^{\sum_{\mu=1}^{\nu-1} n_\mu} | n_1 n_2 \ldots 0_\nu \ldots\rangle$$

$$a^\dagger_\nu | n_1 n_2 \ldots 0_\nu \ldots\rangle = (-1)^{\sum_{\mu=1}^{\nu-1} n_\mu} | n_1 n_2 \ldots 1_\nu \ldots\rangle$$

$$\{a_\mu, a_\nu\} = 0 \qquad \{a^\dagger_\mu, a_\nu\} = \delta_{\mu\nu} \qquad n_\mu = a^\dagger_\mu a_\mu$$

Fock space operators:

$$\hat{F} = \sum_{\mu,\nu} \langle \mu | \hat{f} | \nu\rangle a^\dagger_\mu a_\nu \qquad \hat{F} = \frac{1}{2} \sum_{\mu,\mu',\nu,\nu'} \langle \mu\mu' | \hat{g} | \nu\nu'\rangle a^\dagger_\mu a^\dagger_{\mu'} a_{\nu'} a_\nu$$

Relativistic theory:

metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) = g^{\mu\nu}$

scalar products $a\cdot b = a_\mu b^\mu$

4-vectors: $x^\mu = (ct, \mathbf{x})$, $p^\mu = (E/c, \mathbf{p})$, $A^\mu = (\varphi/c, \mathbf{A})$

derivatives: $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla)$, and $\partial^\mu = \frac{\partial}{\partial x_\mu}$

Dirac gamma-matrices: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4$

Dirac-Pauli representation:

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

m_1	m_2			
J		1		
M		+1		
+1/2	+1/2	1		
J		1	0	
M		0	0	
+1/2	−1/2	1/2	1/2	
−1/2	+1/2	1/2	−1/2	
J		1		
M		−1		
−1/2	−1/2	1		

1×1/2

m_1	m_2		
J		3/2	
M		+3/2	
+1	+1/2	1	
J		3/2	1/2
M		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
J		3/2	1/2
M		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
J		3/2	
M		−3/2	
−1	−1/2	1	

2×1

m_1	m_2			
J		3		
M		+3		
+2	+1	1		
J		3	2	
M		+2	+2	
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
J		3	2	1
M		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10
J		3	2	1
M		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
J		3	2	1
M		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
J		3	2	
M		−2	−2	
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
J		3		
M		−3		
−2	−1	1		

1×1

m_1	m_2			
J		2		
M		+2		
+1	+1	1		
J		2	1	
M		+1	+1	
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
J		2	1	0
M		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
J		2	1	
M		−1	−1	
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
J		2		
M		−2		
−1	−1	1		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

m_1	m_2		
J		5/2	
M		+5/2	
+2	+1/2	1	
J		5/2	3/2
M		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
J		5/2	3/2
M		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
J		5/2	3/2
M		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
J		5/2	3/2
M		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
J		5/2	
M		−5/2	
−2	−1/2	1	

3/2×1/2

m_1	m_2		
J		2	
M		+2	
+3/2	+1/2	1	
J		2	1
M		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
J		2	1
M		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
J		2	1
M		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
J		2	
M		−2	
−3/2	−1/2	1	

3/2×1

m_1	m_2			
J		5/2		
M		+5/2		
+3/2	+1	1		
J		5/2	3/2	
M		+3/2	+3/2	
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
J		5/2	3/2	1/2
M		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
J		5/2	3/2	1/2
M		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
J		5/2	3/2	
M		−3/2	−3/2	
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
J		5/2		
M		−5/2		
−3/2	−1	1		

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

3/2×3/2

m_1	m_2				
J		3			
M		+3			
+3/2	+3/2	1			
J		3	2		
M		+2	+2		
+3/2	+1/2	1/2	1/2		
+1/2	+3/2	1/2	−1/2		
J		3	2	1	
M		+1	+1	+1	
+3/2	−1/2	1/5	1/2	3/10	
+1/2	+1/2	3/5	0	−2/5	
−1/2	+3/2	1/5	−1/2	3/10	
J		3	2	1	0
M		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4
J		3	2	1	
M		−1	−1	−1	
+1/2	−3/2	1/5	1/2	3/10	
−1/2	−1/2	3/5	0	−2/5	
−3/2	+1/2	1/5	−1/2	3/10	
J		3	2		
M		−2	−2		
−1/2	−3/2	1/2	1/2		
−3/2	−1/2	1/2	−1/2		
J		3			
M		−3			
−3/2	−3/2	1			

2×3/2

m_1	m_2				
J		7/2			
M		+7/2			
+2	+3/2	1			
J		7/2	5/2		
M		+5/2	+5/2		
+2	+1/2	3/7	4/7		
+1	+3/2	4/7	−3/7		
J		7/2	5/2	3/2	
M		+3/2	+3/2	+3/2	
+2	−1/2	1/7	16/35	2/5	
+1	+1/2	4/7	1/35	−2/5	
0	+3/2	2/7	−18/35	1/5	
J		7/2	5/2	3/2	1/2
M		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10
J		7/2	5/2	3/2	1/2
M		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5
J		7/2	5/2	3/2	
M		−3/2	−3/2	−3/2	
0	−3/2	2/7	18/35	1/5	
−1	−1/2	4/7	−1/35	−2/5	
−2	+1/2	1/7	−16/35	2/5	
J		7/2	5/2		
M		−5/2	−5/2		
−1	−3/2	4/7	3/7		
−2	−1/2	3/7	−4/7		
J		7/2			
M		−7/2			
−2	−3/2	1			

2×2

m_1	m_2					
J		4				
M		+4				
+2	+2	1				
J		4	3			
M		+3	+3			
+2	+1	1/2	1/2			
+1	+2	1/2	−1/2			
J		4	3	2		
M		+2	+2	+2		
+2	0	3/14	1/2	2/7		
+1	+1	4/7	0	−3/7		
0	+2	3/14	−1/2	2/7		
J		4	3	2	1	
M		+1	+1	+1	+1	
+2	−1	1/14	3/10	3/7	1/5	
+1	0	3/7	1/5	−1/14	−3/10	
0	+1	3/7	−1/5	−1/14	3/10	
−1	+2	1/14	−3/10	3/7	−1/5	
J		4	3	2	1	0
M		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5
J		4	3	2	1	
M		−1	−1	−1	−1	
+1	−2	1/14	3/10	3/7	1/5	
0	−1	3/7	1/5	−1/14	−3/10	
−1	0	3/7	−1/5	−1/14	3/10	
−2	+1	1/14	−3/10	3/7	−1/5	
J		4	3	2		
M		−2	−2	−2		
0	−2	3/14	1/2	2/7		
−1	−1	4/7	0	−3/7		
−2	0	3/14	−1/2	2/7		
J		4	3			
M		−3	−3			
−1	−2	1/2	1/2			
−2	−1	1/2	−1/2			
J		4				
M		−4				
−2	−2	1				

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.