Exam (Tentti): 5 problems, 4 hours

1. As you remember, in the quantum mechanical description of elastic scattering the Born approximation for the scattering amplitude is

$$f_B(\theta,\varphi) = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int d^3r' e^{-i(\mathbf{k_f}-\mathbf{k_i})\cdot\mathbf{r}'}V(\mathbf{r}')$$

where $\mathbf{k_i} = k\hat{\mathbf{e}}_z$, $\mathbf{k_f} = k\mathbf{r}/r = k\hat{\mathbf{e}}_r$ and $k^2 = 2\mu E/\hbar^2$. Let's consider the following potential:

$$V(r) = \begin{cases} V_0, & r \leq a \\ 0, & r > a \end{cases}$$

where V_0 is a constant.

a) Compute $f_B(\theta,\varphi)$ in this case. In the end, write your result in a form which shows the dependence of the obtained scattering amplitude on the scattering angles and energy.

b) What is $f_B(\theta,\varphi)$ for backward scattering off this potential in the high-energy limit ($ka \to \infty$)?

c) What is the elastic total cross section in the low-energy limit ($ka \ll 1$) in the case of the above potential, according to the Born approximation?

2. A spinless hydrogen atom, which is in the 2p state $|n{=}2, l{=}1, m{=}0\rangle$, is put into a time-dependent perturbing potential

$$\hat{V}_S(t) = C\frac{\hat{z}^2}{t^2+\tau^2},$$

where C and $\tau > 0$ are real constants and $\hat{z}$ is the z-coordinate operator.

Using the Gaunt's formula and the attached table of Clebsch-Gordan coefficients, calculate the probability of a transition from the state $|2\,1\,0\rangle$ to the highest-l $n{=}4$ state $|4\,l\,0\rangle$ which is allowed according to the lowest-order time-dependent perturbation theory during an infinitely long period of time (set $t_0 \to -\infty$ and $t \to \infty$).

3. As you remember, a rotation of a state vector – formulated in terms of the standard Euler angles α, β and γ – is caused by the operator

$$\hat{D}(\alpha,\beta,\gamma) = e^{-\frac{i}{\hbar}\alpha\hat{J}_z}e^{-\frac{i}{\hbar}\beta\hat{J}_y}e^{-\frac{i}{\hbar}\gamma\hat{J}_z}$$

Let's consider a one-electron system where the orbital angular momentum $\mathbf{L} = 0$. Suppose that the system is originally in a state which is an eigenstate of the squared total angular momentum $\mathbf{J}^2$ and where the z-component of the electron's spin is up, $+\frac{\hbar}{2}$. What is the state of the system after the above rotation? Express your result in the basis $|j,m\rangle$ and in terms of coefficient functions which depend on α, β and γ. In such a rotated state, what is the probability for finding the electron still with spin up?

4. Let's consider a nonrelativistic gas of N identical noninteracting spin-$\frac{3}{2}$ fermions, whose Hamilton operator in terms of 1-particle operators is

$$\hat{H} = \sum_{i=1}^{N} \hat{H}_i^{(1)} = \sum_{i=1}^{N} \frac{\hat{\mathbf{p}}_i^2}{2m}.$$

The 1-particle wave functions, when putting the free particles in a box of volume $V = L^3$ and requiring periodic boundary conditions, are known to be

$$\phi_{\mathbf{p},s_z}(\mathbf{x},\sigma) = \langle \mathbf{x},\sigma|\mathbf{p},s_z\rangle = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \delta_{\sigma s_z}$$

with $\langle \mathbf{p}_1, s_{1z}|\mathbf{p}_2, s_{2z}\rangle = \delta_{\mathbf{p}_1\mathbf{p}_2}\delta_{s_{1z}s_{2z}}$ and discrete momentum values $\mathbf{p} = \frac{2\pi\hbar}{L}(n_x, n_y, n_z)$ where $n_i = 0, \pm 1, \pm 2, \ldots$.

a) What is the ground state energy of such a system when $N = 76$?

b) Using the Fock-space formalism, compute the single-particle correlation function for such a fermion gas,

$$G_\sigma(\mathbf{x}-\mathbf{x}') \equiv \frac{4}{n}\langle F|\psi^\dagger(\mathbf{x},\sigma)\psi(\mathbf{x}',\sigma)|F\rangle$$

in the limit $N, V \to \infty$, keeping the average particle density, $n = N/V$, constant. Consult the collection of formulae for the field operators. The ground state is

$$|F\rangle = \prod_{\substack{\mathbf{p},s_z \\ |\mathbf{p}|\le p_F}} a^\dagger_{\mathbf{p},s_z}|0\rangle,$$

and the Fermi momentum $p_F = \hbar k_F = \hbar(\frac{3}{2}n\pi^2)^{1/3}$. In the continuum limit $V \to \infty$ use $\rho(\mathbf{p}) = \frac{V}{(2\pi\hbar)^3}$ as the density of the momentum states. You should be able to identify the spherical Bessel function j_1 in your answer and show that the answer depends only on $k_F|\mathbf{x}-\mathbf{x}'|$.

5. Starting from the Lorentz-covariant form of the Dirac equation (DE) for a spin-$\frac{1}{2}$ particle in classical electromagnetic field,

$$[\gamma^\mu(i\hbar\partial_\mu - qA_\mu(x)) - mc]\Psi(x) = 0,$$

show that for the stationary case with time-independent weak electromagnetic field the nonrelativistic (NR) limit of this equation is the Pauli equation,

$$\left[\frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A}(\mathbf{x}))^2\mathbf{1}_2 - \frac{q\hbar}{2m}\vec{\sigma}\cdot\mathbf{B}(\mathbf{x}) + q\varphi(\mathbf{x})\mathbf{1}_2\right]\psi_{NR}(\mathbf{x}) = E_{NR}\psi_{NR}(\mathbf{x}).$$

Hints: First bring the DE into the form $i\hbar\partial_0\Psi = \ldots$, then use the ansatz

$$\Psi(x) = e^{-\frac{i}{\hbar}Et}\begin{pmatrix}\psi_u(\mathbf{x})\\ \psi_l(\mathbf{x})\end{pmatrix}$$

and the Dirac-Pauli representation. Recall also that $A^\mu = (\frac{\varphi}{c}, \mathbf{A})$ and $\hat{\mathbf{p}} = -i\hbar\nabla$.

Collection of formulae:

Spherical coordinates and spherical harmonics:

$$\mathbf{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \qquad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{1}{\hbar^2 r^2}\hat{L}^2$$

$$\int d^3r = \int_0^\infty drr^2 \int_{4\pi} d\Omega = \int_0^\infty drr^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi = \int_0^\infty drr^2 \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\varphi$$

$$\hat{L}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\theta,\varphi) \qquad \hat{L}_z Y_{lm}(\theta,\varphi) = \hbar m Y_{lm}(\theta,\varphi)$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad \int d\Omega\, Y^*_{l'm'}(\theta,\varphi)Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

$$Y_{lm}(\theta,\varphi) = (-1)^{\frac{m+|m|}{2}}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)e^{im\varphi} \qquad Y_{l,-m}(\theta,\varphi) = (-1)^m Y^*_{l,m}(\theta,\varphi)$$

$$P_l^k(z) = (1-z^2)^{k/2}\frac{d^k}{dz^k}P_l(z) \qquad P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l$$

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$$

$$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \qquad Y_{2\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\varphi} \qquad Y_{2\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$$

Stationary Schrödinger equation, the radial part:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1) - r^2\frac{2m}{\hbar^2}V(r)\right]R(r) = 0, \qquad k^2 = \frac{2mE}{\hbar^2}$$

Spherical Bessel & Neumann functions:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1)\right]R(r) = 0 \to \qquad R(r) = Aj_l(kr) + Bn_l(kr)$$

$$j_l(x) = 2^l x^l \sum_{s=0}^\infty \frac{(-1)^s(s+l)!}{s!(2s+2l+1)!}x^{2s} \qquad n_l(x) = \frac{(-1)^{l+1}}{2^l x^{l+1}}\sum_{s=0}^\infty \frac{(-1)^s(s-l)!}{s!(2s-2l)!}x^{2s}$$

$$j_0(x) = \frac{\sin x}{x} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_0(x) = -\frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

Transition probability, lowest order, $i \neq f$:

$$P_{fi}(t,t_0) \equiv |\langle\phi_f|\psi(t)\rangle|^2 \approx \frac{1}{\hbar^2}\left|\int_{t_0}^t dt_1\langle\phi_f|\hat{V}_S(t_1)|\phi_i\rangle e^{i(E_f-E_i)t_1/\hbar}\right|^2$$

Power series, Taylor expansions:

$$e^x = \sum_{n=0}^\infty \frac{x^n}{n!} \qquad \cos x = \sum_{n=0}^\infty (-1)^n\frac{x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^\infty (-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots \qquad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

For integrations:

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}, \qquad \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$\mathrm{Res} f(z)\big|_{z=z_0} = \lim_{z\to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} [(z-z_0)^n f(z)] \qquad \oint_C dz f(z) = 2\pi i \sum_{j=1}^{n} \mathrm{Res} f(z)\big|_{z=z_j}$$

Hydrogen-like atom wave-functions:

$$\Psi_{nlm}(\mathbf{x}) = R_{nl}(r) Y_{lm}(\theta,\varphi) \qquad \kappa = \frac{Z}{na} \qquad a = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

$$R_{nl}(r) = \sqrt{(2\kappa)^3 \frac{(n-l-1)!}{2n(n+l)!}} (2\kappa r)^l e^{-\kappa r} L_{n-l-1}^{2l+1}(2\kappa r) \qquad L_p^q(x) = \sum_{k=0}^{p} (-1^k) \frac{(p+q)! x^k}{(p-k)!(q+k)!k!}$$

$$R_{10} = 2\left(\frac{Z}{a}\right)^{3/2} e^{-Zr/a} \qquad R_{20} = \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2}\left(1-\frac{Zr}{2a}\right)e^{-Zr/2a} \qquad R_{21} = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r e^{-Zr/2a}$$

$$R_{30} = \frac{2}{3\sqrt{3}}\left(\frac{Z}{a}\right)^{3/2}\left(1-\frac{2Zr}{3a}+\frac{2}{27}\left(\frac{Zr}{3a}\right)^2\right)e^{-Zr/3a} \qquad R_{31} = \frac{8}{27\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r\left(1-\frac{Zr}{6a}\right)e^{-Zr/3a}$$

$$R_{32} = \frac{4}{81\sqrt{30}}\left(\frac{Z}{a}\right)^{7/2} r^2 e^{-Zr/3a} \qquad R_{40} = \frac{1}{4}\left(\frac{Z}{a}\right)^{3/2}\left(1-\frac{3Zr}{4a}+\frac{1}{8}\left(\frac{Zr}{a}\right)^2-\frac{1}{192}\left(\frac{Zr}{a}\right)^3\right)e^{-Zr/4a}$$

$$R_{41} = \frac{\sqrt{5}}{16\sqrt{3}}\left(\frac{Z}{a}\right)^{5/2} r\left(1-\frac{Zr}{4a}+\frac{1}{80}\left(\frac{Zr}{a}\right)^2\right)e^{-Zr/4a}$$

$$R_{42} = \frac{1}{64\sqrt{5}}\left(\frac{Z}{a}\right)^{7/2} r^2\left(1-\frac{Zr}{12a}\right)e^{-Zr/4a} \qquad R_{43} = \frac{1}{768\sqrt{35}}\left(\frac{Z}{a}\right)^{9/2} r^3 e^{-Zr/4a}$$

Spherical spinors:

$$(\mathcal{Y}_{ljm}(\Omega))_{m_s} = \langle \Omega, m_s | l, s=\frac{1}{2}, j, m\rangle_c \qquad \int d\Omega \mathcal{Y}_{ljm}(\Omega)^\dagger \mathcal{Y}_{l'j'm'}(\Omega) = \delta_{ll'}\delta_{jj'}\delta_{mm'}$$

Trigonometric functions:

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \cos^2 x + \sin^2 x = 1 \quad \sin 2x = 2\sin x\cos x$$

Euler: $e^{i\alpha} = \cos\alpha + i\sin\alpha \qquad \cos\alpha = \frac{1}{2}(e^{i\alpha}+e^{-i\alpha}) \qquad \sin\alpha = \frac{1}{2i}(e^{i\alpha}-e^{-i\alpha})$

Angular momentum:

$$\hat{\mathbf{J}}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \qquad \hat{J}_z|j,m\rangle = \hbar m|j,m\rangle$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \qquad \hat{J}_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$$

$$[\hat{J}_i,\hat{J}_j] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}\hat{J}_k, \quad [\hat{\mathbf{J}}^2,\hat{J}_i] = 0$$

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k \qquad \{\sigma_i,\sigma_j\} = 2\delta_{ij}\mathbf{1}_2 \qquad (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\mathbf{1}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

Gaunt's formula:

$$\int d\Omega\, Y^*_{lm}(\Omega) Y_{l_1 m_1}(\Omega) Y_{l_2 m_2}(\Omega) = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\, {}_u\langle l_1 l_2 m_1 m_2 | l_1 l_2 l m\rangle_c\, {}_u\langle l_1 l_2 0 0 | l_1 l_2 l 0\rangle_c$$

Wigner-Eckart theorem:

$$\langle \xi' j' m' | \hat{T}^{(k)}_q | \xi j m\rangle = \frac{1}{\sqrt{2j'+1}}\, {}_u\langle j k m q | j k j' m'\rangle_c\, \langle \xi' j' || T^{(k)} || \xi j\rangle$$

where $\langle \xi' j' || T^{(k)} || \xi j\rangle \equiv \frac{1}{\sqrt{2j'+1}} \sum_{m_1, m_2, q'} \langle \xi' j' m_1 | \hat{T}^{(k)}_{q'} | \xi j m_2\rangle \langle j k m_2 q' | j k j' m_1\rangle$

$SU(2)$ tensor operator:

$$[\hat{J}_z, \hat{T}^{(k)}_q] = q \hat{T}^{(k)}_q \qquad [\hat{J}_\pm, \hat{T}^{(k)}_q] = \sqrt{k(k+1) - q(q\pm 1)}\, \hat{T}^{(k)}_{q\pm 1},$$

where q refers to the spherical components, which for a vector operator are

$$\hat{V}_{+1} = -\frac{1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y), \qquad \hat{V}_0 = \hat{V}_z \qquad \hat{V}_{-1} = +\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y)$$

Spherical unit vectors:

$$\hat{e}_{\pm 1} = \mp\frac{1}{\sqrt{2}}(\hat{e}_x \pm i\hat{e}_y), \qquad \hat{e}_0 = \hat{e}_z$$

Scalar products in spherical basis: $\mathbf{A}\cdot\mathbf{B} = -A_{+1}B_{-1} - A_{-1}B_{+1} + A_0 B_0$

Fermionic operators in the Fock space:

$$a_\nu | n_1 n_2 \dots 1_\nu \dots\rangle = (-1)^{\sum_{\mu=1}^{\nu-1} n_\mu} | n_1 n_2 \dots 0_\nu \dots\rangle$$

$$a^\dagger_\nu | n_1 n_2 \dots 0_\nu \dots\rangle = (-1)^{\sum_{\mu=1}^{\nu-1} n_\mu} | n_1 n_2 \dots 1_\nu \dots\rangle$$

$$\{a_\mu, a_\nu\} = 0 \qquad \{a^\dagger_\mu, a_\nu\} = \delta_{\mu\nu} \qquad n_\mu = a^\dagger_\mu a_\mu$$

$$\hat{F} = \sum_{\mu,\nu} \langle \mu | \hat{f} | \nu\rangle a^\dagger_\mu a_\nu \qquad \hat{F} = \frac{1}{2}\sum_{\mu,\mu',\nu,\nu'} \langle \mu\mu' | \hat{g} | \nu\nu'\rangle a^\dagger_\mu a^\dagger_{\mu'} a_{\nu'} a_\nu$$

$$\psi(\mathbf{x},\sigma) = \sum_\mu \phi_\mu(\mathbf{x},\sigma) a_\mu \qquad \psi^\dagger(\mathbf{x},\sigma) = \sum_\mu \phi^*_\mu(\mathbf{x},\sigma) a^\dagger_\mu,$$

Relativistic theory:

metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) = g^{\mu\nu}$

scalar products $a\cdot b = a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu$

4-vectors: $x^\mu = (ct, \mathbf{x})$, $p^\mu = (E/c, \mathbf{p})$, $A^\mu = (\varphi/c, \mathbf{A})$

derivatives: $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla)$, and $\partial^\mu = \frac{\partial}{\partial x_\mu}$

Dirac gamma-matrices: $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbf{1}_4$

Dirac-Pauli representation:

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Clebsch-Gordan coefficients, $3j$ symbols and some properties thereof:

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m}}{\sqrt{2j+1}}\,_u\langle j_1 j_2 m_1 m_2 | j_1 j_2 j\ -m\rangle_c$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_2 & m_3 & m_1 \end{pmatrix}$$

$$(-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$

$$= \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$$

$$\sum_{m_1 m_2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta_{j_3 j_3'}\delta_{m_3 m_3'}}{2j_3+1}$$

$$\sum_{j_3 m_3}(2j_3+1)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'}\delta_{m_2 m_2'}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \quad \text{unless } m_1+m_2+m_3=0 \text{ and } \triangle(j_1,j_2,j_3)$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad \text{if } j_1+j_2+j_3 \text{ is odd}$$

$$\begin{pmatrix} j+\frac{1}{2} & j & \frac{1}{2} \\ m & -m-\frac{1}{2} & \frac{1}{2} \end{pmatrix} = (-1)^{j-m-\frac{1}{2}}\sqrt{\frac{j-m+\frac{1}{2}}{(2j+2)(2j+1)}}$$

$$\begin{pmatrix} j+1 & j & 1 \\ m & -m-1 & 1 \end{pmatrix} = (-1)^{j-m-1}\sqrt{\frac{(j-m)(j-m+1)}{(2j+3)(2j+2)(2j+1)}}$$

$$\begin{pmatrix} j+1 & j & 1 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m-1}\sqrt{\frac{(j+m+1)(j-m+1)}{(2j+3)(j+1)(2j+1)}}$$

$$\begin{pmatrix} j & j & 1 \\ m & -m-1 & 1 \end{pmatrix} = (-1)^{j-m}\sqrt{\frac{(j-m)(j+m+1)}{(j+1)(2j+1)(2j)}}$$

$$\begin{pmatrix} j & j & 1 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m}\frac{m}{\sqrt{(2j+1)(j+1)j}}$$

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m}\frac{1}{\sqrt{2j+1}}$$

$$\begin{pmatrix} j & j & 2 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m}\frac{3m^2-j(j+1)}{\sqrt{(2j+3)(j+1)(2j+1)2j(2j-1)}}$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

m_1	m_2	J M	J M	...
m_1	m_2	Coefficients		
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.