Quantum Mechanics II
Mid-exam 2 extra
14.1.2011

(You can answer in English or in Finnish.)

1. Give a brief description of the following notions:
    a. Gauge invariance
    b. Pauli equation
    c. Hyperfine interactions
    d. Zero-point energy
    e. Stimulated emission

2. The Hamilton operator of an atom is

$$H = H_0 + \varsigma \vec{L} \cdot \vec{S} + \mu_0 (\vec{L} + 2\vec{S}) \cdot \vec{B}.$$

    a. Describe briefly the origin of each term.
    b. In a strong magnetic field the atom is described mainly by

$$H_1 = H_0 + \mu_0 (\vec{L} + 2\vec{S}) \cdot \vec{B}.$$

    Show that the states $|L\, S m_L m_S\rangle$ are eigenstates of H_1.

    c. Consider $\xi(r)\vec{L} \cdot \vec{S}$ as a small perturbation. Evaluate the effect of this perturbation to the energies of different eigenstates of H_1.
    d. Sketch the energy levels of the atomic state ${}^2P_{3/2}$ under a strong magnetic field.

3. The wave function of the electron obeys in the presence of an electromagnetic field the equation

$$\left[\gamma_\mu \left(i\partial_\mu - eA_\mu\right) - m\right]\psi = 0. \qquad (1)$$

    a. Explain by which principle this equation is obtained from the Dirac equation of a free electron.
    b. Show that the quantity $j^\mu = \bar{\psi}\gamma^\mu \psi = \psi^\dagger \gamma_0 \gamma^\mu \psi = 0$.
    c. What is the equation (the counterpart of the equation (1)) obeyed by the wave function of the positron?
    d. How are the wave functions of the electron and the positron related to each other?

4. In the case of a free one-electron atom, the operators $H, \vec{J}^2, J_z$ and

$$K = \beta(1 + \vec{\Sigma} \cdot \vec{L})$$

have common eigenfunctions

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}.$$

Here $\Sigma^i = \sigma^{jk}$ (cyclically) and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

a) Show that

$$K = \beta\left(\vec{\Sigma} \cdot \vec{J} - \frac{1}{2}\right)$$

and that the eigenvalues of K are $\kappa = \pm\left(j + \frac{1}{2}\right)$.

b) Show that

$$\vec{L}^2 \psi_A = l_A(l_A + 1)\psi_A,$$

$$\vec{L}^2 \psi_B = l_B(l_B + 1)\psi_B,$$

where $l_A = j - \frac{1}{2}$ and $l_B = j + \frac{1}{2}$ in the case $\kappa = \left(j + \frac{1}{2}\right)$ and $l_A = j + \frac{1}{2}$ and $l_B = j - \frac{1}{2}$ in the case $\kappa = -\left(j + \frac{1}{2}\right)$.

Help: $\gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger$.