Final exam (tentti): 5 problems, 4 hours

Regarding all problems:
Remember the *collection of formulae* in the end of the problem sheet.

1. (a) The asymptotic ($r \to \infty$) solution of the integral equation for potential scattering is known to be

$$\Psi_{\mathbf{k_i}}(\mathbf{r}) = \Phi_{\mathbf{k_i}}(\mathbf{r}) - \frac{e^{\pm ikr}}{r}\frac{1}{4\pi}\int d^3r' e^{\mp i\mathbf{k_f}\cdot\mathbf{r}'}U(\mathbf{r}')\Psi_{\mathbf{k_i}}(\mathbf{r}'),$$

where $U(\mathbf{r}) = \frac{2\mu}{\hbar^2}V(\mathbf{r})$, $\mathbf{k_i} = k\hat{\mathbf{e}}_z$, $\mathbf{k_f} = k\hat{\mathbf{e}}_\mathbf{r}$, $k^2 = 2\mu E/\hbar^2$. Using this, derive the Born approximation for the scattering amplitude $f_B(\theta, \phi)$.

(b) Compute the scattering amplitude $f_B(\theta, \phi)$ and the differential cross-section $d\sigma/d\Omega$ in the Born approximation for a radially symmetric delta-function potential

$$V(r) = \alpha\delta(r - a),$$

where a and α are constants. Note that this delta-function above applies only to the radial distance r but not to the angles θ, φ. Express your final results in terms of the dimensionless constant $\beta \equiv \frac{2\mu\alpha a}{\hbar^2}$, and show the energy and scattering angle dependencies of your result explicitly.

(c) Sketch the behaviour of $d\sigma/d\Omega$ as a function of the scattering angle in the case $ka = 3\pi$.

2. A spinless hydrogen atom, which is in its ground state 1s (i.e. $|1, 0, 0\rangle$), is put into a weak time-dependent external electric field, which points into the z direction:

$$\mathbf{E}(t, \mathbf{r}) = \frac{C\hat{\mathbf{e}}_\mathbf{z}}{t^2 + \tau^2},$$

where C and $\tau > 0$ are constants. This gives rise to a perturbation potential

$$\hat{V}(t) = C\frac{e\hat{z}}{t^2 + \tau^2},$$

where e denotes the electron charge.

(a) Using lowest-order time-dependent perturbation theory, find the selection rules for the quantum numbers n, l and m in transitions from the ground state.

(b) Calculate the probability of transition from the ground state 1s to the state 2p during an infinitely long period of time, setting $t_0 \to -\infty$ and $t \to \infty$.

3. (a) Using the Wigner-Eckart theorem, show that for a vector operator $\hat{\mathbf{V}}$ we have

$$\langle \xi j m | \hat{\mathbf{V}} | \xi j m' \rangle = \frac{\langle \xi j m | \hat{\mathbf{V}} \cdot \hat{\mathbf{J}} | \xi j m \rangle}{\hbar^2 j(j+1)} \langle \xi j m | \hat{\mathbf{J}} | \xi j m' \rangle$$

As an application, let's consider the Zeeman effect on the hydrogen energy levels in the following. Let the perturbation potential be

$$\hat{H}_B = \frac{\beta B}{\hbar}(\hat{L}_z + 2\hat{S}_z)$$

where B is the magnitude of the weak magnetic field which is pointing into the z direction, β is the Bohr magneton, $\hat{\mathbf{L}}$ is the orbital angular momentum and $\hat{\mathbf{S}}$ is the spin angular momentum.

(b) We wish to apply the above result below. For this, we should show first that the operator

$$\hat{\mathbf{M}} = \hat{\mathbf{L}} + 2\hat{\mathbf{S}}$$

is a vector operator. Explain how you would show that $\hat{\mathbf{M}}$ indeed is a vector operator. You do **not** have to verify the required identities here but write them down in the Cartesian components $\hat{M}_x$, $\hat{M}_y$ and $\hat{M}_z$, so that it becomes clear what you would compute, given more time.

(c) Applying perturbation theory in the basis $|nlsjm\rangle$, compute the first-order corrections to the Hydrogen atom's energy levels caused by $\hat{H}_B$.

(d) What is the energy-level splitting for the state $ns_{\frac{1}{2}}$ i.e. when $l = 0$ and $j = \frac{1}{2}$? Sketch a figure of the splitting, and mark the relevant quantum numbers in the figure.

4. Let's consider the Fock space formulation of the angular momentum operator in a system of identical fermions.

   (a) Angular momentum is an additive quantity, so that the generic form of the Fock space 1-particle operator (see the collection of formulae) holds. Using this, show (in sufficient detail) that the total angular momentum operator can be written as

$$\hat{\vec{\mathcal{J}}} = \sum_{\alpha}\sum_{j}\sum_{m,m'} a^{\dagger}_{\alpha jm'} a_{\alpha jm} \langle \alpha jm' | \hat{\vec{j}} | \alpha jm \rangle,$$

   where $\hat{\vec{j}} = (\hat{j}_x, \hat{j}_y, \hat{j}_z)$ is the angular momentum operator for a 1-particle state, and α stands for all remaining quantum numbers needed to spesify the basis.

   (b) Explain briefly why we should expect that $\hat{\vec{\mathcal{J}}}$ commutes with the total number-of-particles operator $\hat{N}$.

   (c) Show that indeed $[\hat{\vec{\mathcal{J}}}, \hat{N}] = 0$.

   (d) Let's then consider the following 2-particle state in the Fock space:

$$|F^{(2)}\rangle = C \sum_{m_1,m_2} \langle j_1 j_2 m_1 m_2 | j_1 j_2 JM \rangle a^{\dagger}_{\alpha_2 j_2 m_2} a^{\dagger}_{\alpha_1 j_1 m_1} |0\rangle$$

   where $\langle j_1 j_2 m_1 m_2 | j_1 j_2 JM \rangle$ is a Clebsch-Gordan coefficient and C is a normalization constant. Show that $|F^{(2)}\rangle$ is an eigenstate of $\hat{\mathcal{J}}_z$ with an eigenvalue $\hbar M$.

5. Starting from the Lorentz-covariant form of the Dirac equation (DE) for a spin-$\frac{1}{2}$ particle in classical electromagnetic field,

$$[\gamma^{\mu}(i\hbar\partial_{\mu} - qA_{\mu}(x)) - mc]\Psi(x) = 0,$$

   show that for the stationary case with time-independent weak electromagnetic field the nonrelativistic (NR) limit of this equation is the Pauli equation,

$$\left[\frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A}(\mathbf{x}))^2 \mathbf{1}_2 - \frac{q\hbar}{2m}\vec{\sigma}\cdot\mathbf{B}(\mathbf{x}) + q\varphi(\mathbf{x})\mathbf{1}_2\right]\psi_{NR}(\mathbf{x}) = E_{NR}\psi_{NR}(\mathbf{x}).$$

   Hints: First bring the DE into the form $i\hbar\partial_0\Psi = ...$, then use the ansatz

$$\Psi(x) = e^{-\frac{i}{\hbar}Et}\begin{pmatrix}\psi_u(\mathbf{x})\\ \psi_l(\mathbf{x})\end{pmatrix}$$

   and the Dirac-Pauli representation. Recall also that $A^{\mu} = (\frac{\varphi}{c}, \mathbf{A})$ and $\hat{\mathbf{p}} = -i\hbar\nabla$.

Collection of formulae:

Spherical coordinates and spherical harmonics:

$$\mathbf{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \qquad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{1}{\hbar^2 r^2}\hat{L}^2$$

$$\int d^3r = \int_0^\infty drr^2 \int_{4\pi} d\Omega = \int_0^\infty drr^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi = \int_0^\infty drr^2 \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\varphi$$

$$\hat{L}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\theta,\varphi) \qquad \hat{L}_z Y_{lm}(\theta,\varphi) = \hbar m Y_{lm}(\theta,\varphi)$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad \int d\Omega\, Y^*_{l'm'}(\theta,\varphi) Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

$$Y_{lm}(\theta,\varphi) = (-1)^{\frac{m+|m|}{2}}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos\theta) e^{im\varphi} \qquad Y_{l,-m}(\theta,\varphi) = (-1)^m Y^*_{l,m}(\theta,\varphi)$$

$$P_l^k(z) = (1-z^2)^{k/2}\frac{d^k}{dz^k}P_l(z) \qquad P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l$$

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$$

$$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \qquad Y_{2\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\varphi} \qquad Y_{2\pm2}(\theta,\varphi)\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$$

Stationary Schrödinger equation, the radial part:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1) - r^2\frac{2m}{\hbar^2}V(r)\right]R(r) = 0, \qquad k^2 = \frac{2mE}{\hbar^2}$$

Spherical Bessel & Neumann functions:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1)\right]R(r) = 0 \rightarrow \quad R(r) = Aj_l(kr) + Bn_l(kr)$$

$$j_l(x) = 2^l x^l \sum_{s=0}^\infty \frac{(-1)^s(s+l)!}{s!(2s+2l+1)!}x^{2s} \qquad n_l(x) = \frac{(-1)^{l+1}}{2^l x^{l+1}}\sum_{s=0}^\infty \frac{(-1)^s(s-l)!}{s!(2s-2l)!}x^{2s}$$

$$j_0(x) = \frac{\sin x}{x} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_0(x) = -\frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

Transition probability, lowest order, $i \neq f$:

$$P_{fi}(t,t_0) \equiv |\langle\phi_f|\psi(t)\rangle|^2 \approx \frac{1}{\hbar^2}\left|\int_{t_0}^t dt_1 \langle\phi_f|\hat{V}_S(t_1)|\phi_i\rangle e^{i(E_f-E_i)t_1/\hbar}\right|^2$$

Power series, Taylor expansions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots \qquad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

For integrations:

$$\int_0^{\infty} dx x^n e^{-ax} = \frac{n!}{a^{n+1}}, \qquad \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$\mathrm{Res} f(z)\big|_{z=z_0} = \lim_{z\to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} [(z-z_0)^n f(z)] \qquad \oint_C dz f(z) = 2\pi i \sum_{j=1}^{n} \mathrm{Res} f(z)\big|_{z=z_j}$$

Hydrogen-like atom wave-functions:

$$\Psi_{nlm}(\mathbf{x}) = R_{nl}(r) Y_{lm}(\theta, \varphi) \qquad \kappa = \frac{Z}{na} \qquad a = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

$$R_{nl}(r) = \sqrt{(2\kappa)^3 \frac{(n-l-1)!}{2n(n+l)!}} (2\kappa r)^l e^{-\kappa r} L_{n-l-1}^{2l+1}(2\kappa r) \qquad L_p^q(x) = \sum_{k=0}^{p} (-1^k) \frac{(p+q)! x^k}{(p-k)!(q+k)!k!}$$

$$R_{10} = 2\left(\frac{Z}{a}\right)^{3/2} e^{-Zr/a} \qquad R_{20} = \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2}\left(1 - \frac{Zr}{2a}\right) e^{-Zr/2a} \qquad R_{21} = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r e^{-Zr/2a}$$

Spherical spinors:

$$(\mathcal{Y}_{ljm}(\Omega))_{m_s} = \langle \Omega, m_s | l, s = \frac{1}{2}, j, m\rangle_c \qquad \int d\Omega \mathcal{Y}_{ljm}(\Omega)^\dagger \mathcal{Y}_{l'j'm'}(\Omega) = \delta_{ll'}\delta_{jj'}\delta_{mm'}$$

Trigonometric functions:

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \cos^2 x + \sin^2 x = 1 \quad \sin 2x = 2 \sin x \cos x$$

Euler: $e^{i\alpha} = \cos\alpha + i \sin\alpha \qquad \cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) \qquad \sin\alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$

Angular momentum:

$$\hat{\mathbf{J}}^2 |j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \qquad \hat{J}_z|j,m\rangle = \hbar m|j,m\rangle$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \qquad \hat{J}_\pm|j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

$$[\hat{J}_i, \hat{J}_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} \hat{J}_k, \quad [\hat{\mathbf{J}}^2, \hat{J}_i] = 0$$

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}_2$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\mathbf{1}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

Wignert-Eckart theorem:

$$\langle \xi' j' m' | \hat{T}_q^{(k)} | \xi j m \rangle = \frac{1}{\sqrt{2j'+1}} {}_u\langle j k m q | j k j' m' \rangle_c \langle \xi' j' || T^{(k)} || \xi j \rangle$$

where

$$\langle \xi' j' || T^{(k)} || \xi j \rangle \equiv \frac{1}{\sqrt{2j'+1}} \sum_{m_1, m_2, q'} \langle \xi' j' m_1 | \hat{T}_{q'}^{(k)} | \xi j m_2 \rangle \langle j k m_2 q' | j k j' m_1 \rangle$$

$SU(2)$ tensor operator:

$$[\hat{J}_z, \hat{T}_q^{(k)}] = q \hat{T}_q^{(k)} \qquad [\hat{J}_\pm, \hat{T}_q^{(k)}] = \sqrt{k(k+1) - q(q \pm 1)} \hat{T}_{q\pm 1}^{(k)},$$

where q refers to the spherical components, which for a vector operator are

$$\hat{V}_{+1} = -\frac{1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y), \qquad \hat{V}_0 = \hat{V}_z \qquad \hat{V}_{-1} = +\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y)$$

Spherical unit vectors:

$$\hat{e}_{\pm 1} = \mp \frac{1}{\sqrt{2}}(\hat{e}_x \pm i\hat{e}_y), \qquad \hat{e}_0 = \hat{e}_z$$

Scalar products in spherical basis: $\mathbf{A} \cdot \mathbf{B} = -A_{+1}B_{-1} - A_{-1}B_{+1} + A_0 B_0$

Fermionic operators in the Fock space:

$$a_\nu | n_1 n_2 \ldots 1_\nu \ldots \rangle = (-1)^{\sum_{\mu=1}^{\nu-1} n_\mu} | n_1 n_2 \ldots 0_\nu \ldots \rangle$$

$$a_\nu^\dagger | n_1 n_2 \ldots 0_\nu \ldots \rangle = (-1)^{\sum_{\mu=1}^{\nu-1} n_\mu} | n_1 n_2 \ldots 1_\nu \ldots \rangle$$

$$\{a_\mu, a_\nu\} = 0 \qquad \{a_\mu^\dagger, a_\nu\} = \delta_{\mu\nu} \qquad n_\mu = a_\mu^\dagger a_\mu$$

Fock space operators:

$$\hat{F} = \sum_{\mu,\nu} \langle \mu | \hat{f} | \nu \rangle a_\mu^\dagger a_\nu \qquad \hat{F} = \frac{1}{2} \sum_{\mu,\mu',\nu,\nu'} \langle \mu\mu' | \hat{g} | \nu\nu' \rangle a_\mu^\dagger a_{\mu'}^\dagger a_{\nu'} a_\nu$$

Relativistic theory:

metric tensor $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) = g^{\mu\nu}$

scalar products $a \cdot b = a_\mu b^\mu$

4-vectors: $x^\mu = (ct, \mathbf{x})$, $p^\mu = (E/c, \mathbf{p})$, $A^\mu = (\varphi/c, \mathbf{A})$

derivatives: $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla)$, and $\partial^\mu = \frac{\partial}{\partial x_\mu}$

Dirac gamma-matrices: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4$

Dirac-Pauli representation:

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$