Quantum Mechanics II
Mid-exam 2
17.12.2010

(You can answer in English or in Finnish.)

1. Give a brief description of the following notions:
   a. Creation and annihilation operators
   b. Minimal substitution
   c. Lande's g-factor
   d. Coupled basis of angular momenta
   e. Polarization vector

2. The Hamilton operator of an atom is

$$H = H_0 + \varsigma \vec{L} \cdot \vec{S} + \mu_0(\vec{L} + 2\vec{S}) \cdot \vec{B}.$$

   a. Describe briefly the origin of each term.
   b. In a strong magnetic field the atom is described mainly by

$$H_1 = H_0 + \mu_0(\vec{L} + 2\vec{S}) \cdot \vec{B}.$$

   Show that the states $|L\, S m_L m_S\,\rangle$ are eigenstates of H_1.

   c. Consider $\xi(r)\vec{L} \cdot \vec{S}$ as a small perturbation. Evaluate the effect of this perturbation to the energies of different eigenstates of H_1.
   d. Sketch the energy levels of the atomic state ${}^2P_{3/2}$ under a strong magnetic field.

3.
   a. Show that the Maxwell equations in vacuum,

$$\vec{\nabla} \cdot \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t},$$

   where $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial}{\partial t}\vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$, are invariant under the gauge transformation

$$\vec{A} \to \vec{A} - \vec{\nabla}\Lambda,\ \varphi \to \varphi + \frac{\partial}{\partial t}\Lambda.$$

   b. Show that this gauge invariance allows one to assume $\vec{\nabla} \cdot \vec{A} = 0$.
   c. Show that the electromagnetic field has only two possible polarization states.

4. In the case of a free one-electron atom, the operators $H, \vec{J}^2, J_z$ and

$$K = \beta(1 + \vec{\Sigma} \cdot \vec{L})$$

have common eigenfunctions

$$\Psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}.$$

Here $\Sigma^i = \sigma^{jk}$ (cyclically) and $\sigma^{\mu\upsilon} = \frac{i}{2}[\gamma^\mu, \gamma^\upsilon]$.

a) Show that

$$K = \beta\left(\vec{\Sigma} \cdot \vec{J} - \frac{1}{2}\right)$$

and that the eigenvalues of K are $\kappa = \pm\left(j + \frac{1}{2}\right)$.

b) Show that

$$\vec{L}^2 \psi_A = l_A(l_A + 1)\psi_{A,}$$

$$\vec{L}^2 \psi_B = l_B(l_B + 1)\psi_B,$$

where $l_A = j + \frac{1}{2}$ and $l_B = j - \frac{1}{2}$ in the case $\kappa = \left(j + \frac{1}{2}\right)$ and $l_A = j - \frac{1}{2}$ and $l_B = j + \frac{1}{2}$ in the case $\kappa = -\left(j + \frac{1}{2}\right)$.