1. Let us consider a rotation $R_{\mathbf{n}}(\alpha)$ around an axis $\mathbf{n}$ by an angle α. The effect of the rotation on the position vectors is

$$\mathbf{x} \xrightarrow{R} \mathbf{x}' = R_{\mathbf{n}}(\alpha)\mathbf{x}$$

and on the state vectors

$$|\psi\rangle \xrightarrow{R} |\psi'\rangle = \hat{D}^{(j)}(R_{\mathbf{n}}(\alpha))|\psi\rangle, \qquad \text{where} \qquad \hat{D}^{(j)}(R_{\mathbf{n}}(\alpha)) = e^{-\frac{i}{\hbar}\alpha\mathbf{n}\cdot\hat{\mathbf{J}}}|\psi\rangle.$$

   (a) (3 p.) Find the rotation matrix $\mathbb{D}^{(j)}(R_{\mathbf{n}}(\alpha))$ for $j = \frac{1}{2}$ in the eigenbasis $|j, m\rangle$ of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$, when the rotation is around the x-axis, $\mathbf{n} = \hat{\mathbf{e}}_{\mathbf{x}}$.

   (b) (3 p.) Suppose that the system is originally in the state $|\psi\rangle = |j = \frac{1}{2}, m = \frac{1}{2}\rangle$. In which state is it after the above rotation around the x-axis? What happens if $\alpha = 2\pi$, is the state of the system after the rotation the same as initially?

2. Suppose that the Hamilton operator in the Schrödinger picture is

$$\hat{H}_S(t) = \hat{H}_0 + \hat{V}_S(t),$$

where $\hat{H}_0$ does not depend on time but $\hat{V}_S$ does. In the interaction picture (I-picture), the states are defined in terms of the states in the Schrödinger picture as

$$|\psi(t)\rangle_I \equiv e^{\frac{i}{\hbar}\hat{H}_0 t}|\psi(t)\rangle_S.$$

   (a) (2 p.) Derive the following equation of motion for the states in the I-picture:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle_I = \hat{V}_I(t)|\psi(t)\rangle_I.$$

   (b) (2 p.) Show next that the time-evolution operator $\hat{U}_I(t, t_0)$ in the I-picture satisfies the following integral equation,

$$\hat{U}_I(t, t_0) = \hat{\mathbf{1}} - \frac{i}{\hbar}\int_{t_0}^{t} dt' V_I(t')\hat{U}_I(t', t_0).$$

   (c) (2 p.) Suppose then that the time-dependent potential $V_S(t)$ is a weak perturbation. Using the I-picture, derive the following lowest-order time-dependent perturbation theory result for the transition probability for a transition from an initial state $|\phi_i\rangle$ at t_0 to a state $|\phi_f\rangle$ at t:

$$P_{fi}(t, t_0) \equiv |\langle\phi_f|\psi(t)\rangle|^2 \approx \frac{1}{\hbar^2}\left|\int_{t_0}^{t} dt_1 \langle\phi_f|\hat{V}_S(t_1)|\phi_i\rangle e^{i(E_f - E_i)t_1/\hbar}\right|^2,$$

   where the initial and final states are eigenstates of $\hat{H}_0$: $\hat{H}_0|\phi_f\rangle = E_f|\phi_f\rangle$ and $\hat{H}_0|\phi_i\rangle = E_i|\phi_i\rangle$, and $|\phi_i\rangle \neq |\phi_f\rangle$. We also assume here that the energy spectrum is discrete and nondegenerate.

3. (a) (3 p.) The scattering amplitude for elastic scattering off a potential $V(\mathbf{r})$ is given by

$$f_k(\theta,\phi) = -2\pi^2 \int d^3r' \Phi^*_{\mathbf{k_f}}(\mathbf{r}')U(\mathbf{r}')\Psi_{\mathbf{k_i}}(\mathbf{r}'),$$

where $\mathbf{k_f}$ is the wave vector of the particle after the scattering and $\mathbf{k_i}$ is the wave vector before the scattering, $|\mathbf{k_f}| = |\mathbf{k_i}| = k = \sqrt{2\mu E/\hbar^2}$, and $\Phi_{\mathbf{k_f}}(\mathbf{r}') = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k_f}\cdot\mathbf{r}'}$, and $U(\mathbf{r}) = \frac{2\mu}{\hbar^2}V(\mathbf{r})$. The wave function $\Psi_{\mathbf{k_i}}(\mathbf{r})$ is obtained from the integral equation for the potential scattering,

$$\Psi_{\mathbf{k}}(\mathbf{r}) = \Phi_{\mathbf{k}}(\mathbf{r}) + \int d^3r' G_{\mathbf{k}}(\mathbf{r}-\mathbf{r}')U(\mathbf{r}')\Psi_{\mathbf{k}}(\mathbf{r}'),$$

where $G_{\mathbf{k}}(\mathbf{r}-\mathbf{r}')$ is the Green's function for the operator $\nabla^2 + k^2$ (explicit form of $G_{\mathbf{k}}$ is not needed now).

Form the Born series for the scattering amplitude (write only the two first terms explicitly), and show that in the Born approximation the scattering amplitude is essentially a Fourier transform of the potential,

$$f_B(\theta,\phi) = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int d^3r e^{i(\mathbf{k_i}-\mathbf{k_f})\cdot\mathbf{r}} V(\mathbf{r}).$$

(b) (3 p.) Thomson's plum puddig model of the atoms was excluded by Rutherford's experiment. Assume the distribution

$$\rho_N(\mathbf{x}) = \frac{Z}{(2\pi a_B)^{3/2}} e^{-\mathbf{x}^2/(2a_B^2)} \tag{1}$$

of the positive charge in the atom instead of a point-like nucleus $\rho_N(\vec{x}) = Z\delta(\vec{x})$. Calculate the scattering cross section of the alpha particle off the atom using the Born approximation. You may neglect the electrons which are so light that they are thrown out by the bulldozing alpha particle anyway. Compare your result to that with the point-like nucleus:

$$\frac{d\sigma}{d\Omega}_{\text{point}} = \frac{(2m)^2(ZZ'e^2)^2}{16(\hbar k)^2\sin^4(\theta/2)}. \tag{2}$$

4. (a) (3 p.) Using the Wigner-Eckart theorem, show that for a vector operator $\hat{\mathbf{V}}$, we have

$$\langle \xi jm|\hat{\mathbf{V}}|\xi jm'\rangle = \frac{\langle \xi jm|\hat{\mathbf{V}}\cdot\hat{\mathbf{J}}|\xi jm\rangle}{\hbar^2 j(j+1)}\langle \xi jm|\hat{\mathbf{J}}|\xi jm'\rangle$$

(b) (3 p.) Apply the above result in the following nonrelativistic nuclear shell-model problem, where the system(=nucleus) consists of protons and neutrons (i.e. spin-$\frac{1}{2}$ particles) and their mutual interactions. For those nuclei whose number of protons+neutrons is odd, the oribital angular momentum $\mathbf{L}$ and spin angular momentum $\mathbf{S}$ can be assumed to be those of the last (highest-energy), odd, proton or neutron. Then $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and thus $j = l \pm \frac{1}{2}$, and the quantum numbers in ξ above include l and $s = \frac{1}{2}$. The magnetic moment operator $\hat{\mathbf{M}}$ of the nucleus is defined as

$$\hat{\mathbf{M}} = \gamma_L \hat{\mathbf{L}}/\hbar + \gamma_S \hat{\mathbf{S}}/\hbar,$$

where γ_L and γ_S are constants. The magnetic moment μ of the nucleus is defined as the largest possible absolute value (norm) of the vector $\langle \mathbf{M}\rangle \equiv \langle \xi jm|\hat{\mathbf{M}}|\xi jm\rangle$. Show that for $l = j - \frac{1}{2}$ the magnetic moment μ becomes

$$\mu = (j - \frac{1}{2})\gamma_L + \frac{1}{2}\gamma_S.$$

Collection of formulas:

Fourier transform:

$$\begin{aligned}\tilde{f}(\mathbf{p}) &= \int \frac{d^3x}{\sqrt{(2\pi)^3}} e^{-i\mathbf{p}\cdot\mathbf{x}} f(\mathbf{x}) \\ f(\mathbf{x}) &= \int \frac{d^3p}{\sqrt{(2\pi)^3}} e^{i\mathbf{p}\cdot\mathbf{x}} \tilde{f}(\mathbf{p})\end{aligned}$$

Under suitable assumptions for functions $f(\mathbf{x})$ ja $g(\mathbf{x})$ we have:

$$\int \frac{d^3x}{\sqrt{(2\pi)^3}} f(\mathbf{x}) \nabla^2 g(\mathbf{x}) = \int \frac{d^3x}{\sqrt{(2\pi)^3}} (\nabla^2 f(\mathbf{x})) g(\mathbf{x}).$$

$$g(\mathbf{x}) = \frac{1}{(2\pi\sigma)^{3/2}} e^{-\mathbf{x}^2/(2\sigma^2)} \Rightarrow \tilde{g}(\mathbf{k}) = e^{-\sigma^2\mathbf{k}^2} \tag{3}$$

Spherical coordinates and spherical harmonics:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{1}{\hbar^2 r^2}\hat{L}^2 \qquad d^3r = r^2 dr d\Omega = r^2 dr \sin\theta d\theta d\varphi \qquad \int d\Omega = 4\pi$$

$$\hat{L}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\theta,\varphi) \qquad \hat{L}_z Y_{lm}(\theta,\varphi) = \hbar m Y_{lm}(\theta,\varphi)$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad \int d\Omega\, Y^*_{l'm'}(\theta,\varphi) Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

$$Y_{lm}(\theta,\varphi) = (-1)^{\frac{m+|m|}{2}} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos\theta) e^{im\varphi} \qquad Y_{l,-m}(\theta,\varphi) = (-1)^m Y^*_{l,m}(\theta,\varphi)$$

$$P_l^k(z) = (1-z^2)^{k/2}\frac{d^k}{dz^k}P_l(z) \qquad P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l$$

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$$

$$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \qquad Y_{2\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\varphi} \qquad Y_{2\pm2}(\theta,\varphi)\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$$

Hydrogen-like atom wave-functions:

$$\Psi_{nlm}(\mathbf{x}) = R_{nl}(r) Y_{lm}(\theta,\varphi) \qquad \kappa = \frac{Z}{na} \qquad a = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

$$R_{nl}(r) = \sqrt{(2\kappa)^3\frac{(n-l-1)!}{2n(n+l)!}}(2\kappa r)^l e^{-\kappa r} L^{2l+1}_{n-l-1}(2\kappa r) \qquad L_p^q(x) = \sum_{k=0}^{p}(-1^k)\frac{(p+q)!x^k}{(p-k)!(q+k)!k!}$$

$$R_{10} = 2\left(\frac{Z}{a}\right)^{3/2} e^{-Zr/a} \qquad R_{20} = \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2}\left(1 - \frac{Zr}{2a}\right)e^{-Zr/2a} \qquad R_{21} = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r e^{-Zr/2a}$$

Spherical Bessel & Neumann functions:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1)\right] R(r) = 0 \quad \rightarrow \quad R(r) = Aj_l(kr) + Bn_l(kr)$$

$$j_l(x) = 2^l x^l \sum_{s=0}^{\infty} \frac{(-1)^s (s+l)!}{s!(2s+2l+1)!} x^{2s} \qquad n_l(x) = \frac{(-1)^{l+1}}{2^l x^{l+1}} \sum_{s=0}^{\infty} \frac{(-1)^s (s-l)!}{s!(2s-2l)!} x^{2s}$$

$$j_0(x) = \frac{\sin x}{x} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_0(x) = -\frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

For integrations:

$$\int_0^{\infty} dx x^n e^{-ax} = \frac{n!}{a^{n+1}}, \qquad \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \qquad \mathrm{Res} f(z)\big|_{z=z_0} = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}[(z-z_0)^n f(z)]$$

$$\oint_C dz f(z) = 2\pi i \sum_{j=1}^{n} \mathrm{Res} f(z)\big|_{z=z_j}.$$

Trigonometry: $\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos^2 x + \sin^2 x = 1$

Angular momentum:

$$\hat{\mathbf{J}}^2 |j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \qquad \hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \qquad \hat{J}_\pm |j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$$

$$[\hat{J}_i, \hat{J}_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} \hat{J}_k, \quad [\hat{\mathbf{J}}^2, \hat{J}_i] = 0$$

Power series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}_2$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\mathbf{1}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

Wignert-Eckart theorem:

$$\langle \xi' j' m' | \hat{T}_q^{(k)} | \xi j m \rangle = \frac{1}{\sqrt{2j'+1}} {}_u\langle jkmq | jkj'm' \rangle_c \langle \xi' j' || T^{(k)} || \xi j \rangle$$

where

$$\langle \xi' j' || T^{(k)} || \xi j \rangle \equiv \frac{1}{\sqrt{2j'+1}} \sum_{m_1, m_2, q'} \langle \xi' j' m_1 | \hat{T}^{(k)}_{q'} | \xi j m_2 \rangle \langle j k m_2 q' | j k j' m_1 \rangle$$

$SU(2)$ tensor operator:

$$[\hat{J}_z, \hat{T}^{(k)}_q] = q \hat{T}^{(k)}_q \qquad [\hat{J}_\pm, \hat{T}^{(k)}_q] = \sqrt{k(k+1) - q(q \pm 1)} \hat{T}^{(k)}_{q \pm 1},$$

where q refers to the spherical components, which for a vector operator are

$$\hat{V}_{+1} = -\frac{1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y), \qquad \hat{V}_0 = \hat{V}_z \qquad \hat{V}_{-1} = +\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y)$$

Spherical unit vectors

$$\hat{e}_{\pm 1} = \mp \frac{1}{\sqrt{2}}(\hat{e}_x \pm i\hat{e}_y), \qquad \hat{e}_0 = \hat{e}_z$$

Scalar products in spherical basis: $\mathbf{A} \cdot \mathbf{B} = -A_{+1}B_{-1} - A_{-1}B_{+1} + A_0 B_0$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

m_1 m_2	J M	J M	...
m_1 m_2	Coefficients		
. .			

$1/2\times1/2$

m_1 m_2	$J=1$, $M=+1$
+1/2 +1/2	1

m_1 m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=1$, $M=-1$
−1/2 −1/2	1

$1\times1/2$

m_1 m_2	$J=3/2$, $M=+3/2$
+1 +1/2	1

m_1 m_2	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	$J=3/2$, $M=-3/2$
−1 −1/2	1

2×1

m_1 m_2	$J=3$, $M=+3$
+2 +1	1

m_1 m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	$J=3$, $M=-3$
−2 −1	1

1×1

m_1 m_2	$J=2$, $M=+2$
+1 +1	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	$J=2$, $M=-2$
−1 −1	1

$3/2\times1$

m_1 m_2	$J=5/2$, $M=+5/2$
+3/2 +1	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-5/2$
−3/2 −1	1

$2\times1/2$

m_1 m_2	$J=5/2$, $M=+5/2$
+2 +1/2	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	$J=5/2$, $M=-5/2$
−2 −1/2	1

$3/2\times1/2$

m_1 m_2	$J=2$, $M=+2$
+3/2 +1/2	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	$J=2$, $M=-2$
−3/2 −1/2	1

$$Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1=-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0=\sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta-\frac{1}{2}\Big)$$

$$Y_2^1=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m}=(-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0}=\sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1j_2m_1m_2|j_1j_2JM\rangle = (-1)^{J-j_1-j_2}\langle j_2j_1m_2m_1|j_2j_1JM\rangle$$

$$d^j_{m',m}=(-1)^{m-m'}d^j_{m,m'}=d^j_{-m,-m'}$$

$$d^1_{0,0}=\cos\theta \qquad d^{1/2}_{1/2,1/2}=\cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2}=-\sin\frac{\theta}{2}$$

$$d^1_{1,1}=\frac{1+\cos\theta}{2} \qquad d^1_{1,0}=-\frac{\sin\theta}{\sqrt2} \qquad d^1_{1,-1}=\frac{1-\cos\theta}{2}$$

$3/2\times3/2$

m_1 m_2	$J=3$, $M=+3$
+3/2 +3/2	1

m_1 m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	$J=3$, $M=-3$
−3/2 −3/2	1

$2\times3/2$

m_1 m_2	$J=7/2$, $M=+7/2$
+2 +3/2	1

m_1 m_2	$J=7/2$, $M=+5/2$	$J=5/2$, $M=+5/2$
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	$J=7/2$, $M=+3/2$	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	$J=7/2$, $M=+1/2$	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	$J=7/2$, $M=-1/2$	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	$J=7/2$, $M=-3/2$	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	$J=7/2$, $M=-5/2$	$J=5/2$, $M=-5/2$
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	$J=7/2$, $M=-7/2$
−2 −3/2	1

2×2

m_1 m_2	$J=4$, $M=+4$
+2 +2	1

m_1 m_2	$J=4$, $M=+3$	$J=3$, $M=+3$
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	$J=4$, $M=+2$	$J=3$, $M=+2$	$J=2$, $M=+2$
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	$J=4$, $M=+1$	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	$J=4$, $M=0$	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	$J=4$, $M=-1$	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	$J=4$, $M=-2$	$J=3$, $M=-2$	$J=2$, $M=-2$
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	$J=4$, $M=-3$	$J=3$, $M=-3$
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	$J=4$, $M=-4$
−2 −2	1

$$d^{3/2}_{3/2,3/2}=\frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2}=-\sqrt3\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2}=\sqrt3\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2}=-\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2}=\frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2}=-\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2}=\Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^2_{2,1}=-\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0}=\frac{\sqrt6}{4}\sin^2\theta$$

$$d^2_{2,-1}=-\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2}=\Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^2_{1,1}=\frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0}=-\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1}=\frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0}=\Big(\frac{3}{2}\cos^2\theta-\frac{1}{2}\Big)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.