2. exam (2. välikoe): 4 problems, 4 hours

Regarding all problems:
Consult the *collection of formulae* in the end of the problem sheet!

1. The SU(2) rotation matrices in the $j = \frac{1}{2}$ representation are known to be given in terms of the Pauli spin matrices, rotation angle and rotation axis as

$$\mathcal{D}^{(\frac{1}{2})} = e^{-i\frac{1}{2}\alpha\mathbf{n}\cdot\vec{\sigma}}.$$

a) Show that $\mathcal{D}^{(\frac{1}{2})} = \mathbf{1}_2 \cos(\alpha/2) - i(\mathbf{n}\cdot\vec{\sigma})\sin(\alpha/2)$

b) Verify that indeed $\mathcal{D}^{(\frac{1}{2})} \in \mathrm{SU}(2)$.

Consider then a matrix $U = \frac{1}{\sqrt{2}}(\mathbf{1}_2 + i\sigma_x)$.

c) Show that $U \in \mathrm{SU}(2)$.

d) Write down the SO(3) rotation matrix R in 3-dimensional coordinate space which corresponds to U above.

e) Show that the matrices $\{\mathbf{1}_2, U^2, U^4, U^6\}$ form a group (i.e. a subgroup of SU(2)).

2. Let's put a spinless hydrogen atom in a weak external constant electric field, i.e. perturb the hydrogen atom with a potential

$$V = e\mathcal{E}\,z = e\mathcal{E}\,r\cos\theta = e\mathcal{E}\sqrt{\tfrac{4\pi}{3}}\,rY_{10}(\theta,\varphi),$$

where the product $e\mathcal{E}$ is a positive constant. Using the Gaunt's formula, consider the Stark effect on a spinless hydrogen atom in the case $n = 2$, i.e. determine how such a perturbation changes the hydrogen energy levels ϵ_2. Sketch also a figure of the obtained energy level splitting. A table of the Clebsch-Gordan coefficients and related material is attached.

3. Let's consider a nonrelativistic gas of N identical noninteracting spin-s fermions, whose Hamilton operator in terms of 1-particle operators is

$$\hat{H} = \sum_{i=1}^{N} \hat{H}_i^{(1)} = \sum_{i=1}^{N} \frac{\hat{\mathbf{p}}_i^2}{2m}.$$

The 1-particle wave functions, when putting the free particles in a box of volume $V = L^3$ and requiring periodic boundary conditions, are known to be

$$\phi_{\mathbf{p},s_z}(\mathbf{x},\sigma) = \langle \mathbf{x},\sigma|\mathbf{p},s_z\rangle = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \delta_{\sigma s_z}$$

with $\langle \mathbf{p}_1, s_{1z}|\mathbf{p}_2, s_{2z}\rangle = \delta_{\mathbf{p}_1\mathbf{p}_2}\delta_{s_{1z}s_{2z}}$ and discrete momentum values $\mathbf{p} = \frac{2\pi\hbar}{L}(n_x, n_y, n_z)$ where $n_i = 0, \pm 1, \pm 2, \ldots$.

a) Form the Hamilton operator of this system in the Fock space. Consult the collection of formulae for the needed generic form of the Fock space operator.

b) The ground state of this system is

$$|F\rangle = \prod_{\substack{\mathbf{p},s_z \\ |\mathbf{p}|\le p_F}} a^\dagger_{\mathbf{p},s_z}|0\rangle.$$

Show explicitly the effect of the total particle number operator $\hat{N}$ (see again the collection of the formulae) on the ground state $|F\rangle$.

c) Compute the Fermi momentum, p_F, of such a gas in the limit $N, V \to \infty$. Express p_F in terms of the particle spin s and the average particle density $n \equiv N/V$. In the continuum limit $V \to \infty$ use $\rho(\mathbf{p}) = \frac{V}{(2\pi\hbar)^3}$ as the density of the momentum states.

d) Compute the single-particle correlation function for our fermion gas,

$$G_\sigma(\mathbf{x}-\mathbf{x}') \equiv \frac{2s+1}{n}\langle F|\psi^\dagger(\mathbf{x},\sigma)\psi(\mathbf{x}',\sigma)|F\rangle$$

in the limit $N, V \to \infty$. For the field operators, consult the collection of formulae. You should also identify the spherical Bessel function j_1 in your answer and show that the answer depends only on $k_F|\mathbf{x}-\mathbf{x}'|$.

4. a) Substituting an ansatz

$$\Psi(x) = u(p)e^{-\frac{i}{\hbar}p\cdot x}$$

into the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi(x) = 0,$$

and using the Clifford algebra for the gamma-matrices, show that the Dirac equation has both positive-energy and negative-energy solutions. Which are the allowed values of energy?

b) Using the Dirac-Pauli representation and considering the positive energy case, find an explicit 4-component form for the momentum-space spinor $u^{(1)}(p)$ which in the limit $\mathbf{p} \to 0$ becomes an eigenspinor of Σ_z with an eigenvalue $+\frac{\hbar}{2}$. You don't have to consider the normalization of $u^{(1)}(p)$. In the end, show explicitly that indeed $\Sigma_z u^{(1)}(\mathbf{p} = 0) = +\frac{\hbar}{2}u^{(1)}(\mathbf{p} = 0)$ for the 4-spinor you obtained.

c) Derive the Dirac Hamilton operator $\hat{H}_D$ from the Dirac equation above, and using the 4-spinor $u^{(1)}(p)$ which you obtained (in its block form), show explicitly that indeed $\hat{H}_D\psi(\mathbf{x}) = E\psi(\mathbf{x})$, where $\psi(\mathbf{x})$ is the stationary part of the spinor $\Psi(x)$ above.

Collection of formulae:

Spherical coordinates and spherical harmonics:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{1}{\hbar^2 r^2}\hat{L}^2 \qquad d^3r = r^2 dr d\Omega = r^2 dr \sin\theta d\theta d\varphi \qquad \int d\Omega = 4\pi$$

$$\hat{L}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\theta,\varphi) \qquad \hat{L}_z Y_{lm}(\theta,\varphi) = \hbar m Y_{lm}(\theta,\varphi)$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad \int d\Omega\, Y^*_{l'm'}(\theta,\varphi) Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

$$Y_{lm}(\theta,\varphi) = (-1)^{\frac{m+|m|}{2}}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos\theta) e^{im\varphi} \qquad Y_{l,-m}(\theta,\varphi) = (-1)^m Y^*_{l,m}(\theta,\varphi)$$

$$P_l^k(z) = (1-z^2)^{k/2}\frac{d^k}{dz^k}P_l(z) \qquad P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l$$

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$$

$$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \qquad Y_{2\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\varphi} \qquad Y_{2\pm2}(\theta,\varphi)\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$$

Transition probability, lowest order, $i \neq f$:

$$P_{fi}(t,t_0) \equiv |\langle\phi_f|\psi(t)\rangle|^2 \approx \frac{1}{\hbar^2}\left|\int_{t_0}^{t} dt_1 \langle\phi_f|\hat{V}_S(t_1)|\phi_i\rangle e^{i(E_f-E_i)t_1/\hbar}\right|^2$$

Hydrogen-like atom wave-functions:

$$\Psi_{nlm}(\mathbf{x}) = R_{nl}(r) Y_{lm}(\theta,\varphi) \qquad \kappa = \frac{Z}{na} \qquad a = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

$$R_{nl}(r) = \sqrt{(2\kappa)^3\frac{(n-l-1)!}{2n(n+l)!}}(2\kappa r)^l e^{-\kappa r} L_{n-l-1}^{2l+1}(2\kappa r) \qquad L_p^q(x) = \sum_{k=0}^{p}(-1^k)\frac{(p+q)!x^k}{(p-k)!(q+k)!k!}$$

$$R_{10} = 2\left(\frac{Z}{a}\right)^{3/2} e^{-Zr/a} \qquad R_{20} = \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2}\left(1-\frac{Zr}{2a}\right)e^{-Zr/2a} \qquad R_{21} = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r e^{-Zr/2a}$$

Spherical Bessel & Neumann functions:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[(kr)^2 - l(l+1)\right]R(r) = 0 \quad \rightarrow \quad R(r) = Aj_l(kr) + Bn_l(kr)$$

$$j_l(x) = 2^l x^l \sum_{s=0}^{\infty}\frac{(-1)^s(s+l)!}{s!(2s+2l+1)!}x^{2s} \qquad n_l(x) = \frac{(-1)^{l+1}}{2^l x^{l+1}}\sum_{s=0}^{\infty}\frac{(-1)^s(s-l)!}{s!(2s-2l)!}x^{2s}$$

$$j_0(x) = \frac{\sin x}{x} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_0(x) = -\frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

For integrations:

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}, \qquad \int_{-\infty}^\infty dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \qquad \mathrm{Res} f(z)\big|_{z=z_0} = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}[(z-z_0)^n f(z)]$$

$$\oint_C dz f(z) = 2\pi i \sum_{j=1}^n \mathrm{Res} f(z)\big|_{z=z_j}.$$

Trigonometry: $\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos^2 x + \sin^2 x = 1$

Angular momentum:

$$\hat{\mathbf{J}}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \qquad \hat{J}_z|j,m\rangle = \hbar m|j,m\rangle$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \qquad \hat{J}_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$$

$$[\hat{J}_i, \hat{J}_j] = i\hbar\sum_{k=1}^3 \epsilon_{ijk}\hat{J}_k, \quad [\hat{\mathbf{J}}^2, \hat{J}_i] = 0$$

Power series:

$$e^x = \sum_{n=0}^\infty \frac{x^n}{n!} \qquad \cos x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}_2$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\mathbf{1}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

Wignert-Eckart theorem:

$$\langle\xi' j' m'|\hat{T}_q^{(k)}|\xi j m\rangle = \frac{1}{\sqrt{2j'+1}}\, {}_u\langle jkmq|jkj'm'\rangle_c\, \langle\xi' j'||T^{(k)}||\xi j\rangle$$

where

$$\langle\xi' j'||T^{(k)}||\xi j\rangle \equiv \frac{1}{\sqrt{2j'+1}} \sum_{m_1,m_2,q'} \langle\xi' j' m_1|\hat{T}_{q'}^{(k)}|\xi j m_2\rangle\langle jkm_2q'|jkj'm_1\rangle$$

$SU(2)$ tensor operator:

$$[\hat{J}_z, \hat{T}_q^{(k)}] = q\hat{T}_q^{(k)} \qquad [\hat{J}_\pm, \hat{T}_q^{(k)}] = \sqrt{k(k+1)-q(q\pm 1)}\hat{T}_{q\pm 1}^{(k)},$$

where q refers to the spherical components, which for a vector operator are

$$\hat{V}_{+1} = -\frac{1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y), \qquad \hat{V}_0 = \hat{V}_z \qquad \hat{V}_{-1} = +\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y)$$

Spherical unit vectors:

$$\hat{e}_{\pm 1} = \mp \frac{1}{\sqrt{2}}(\hat{e}_x \pm i\hat{e}_y), \qquad \hat{e}_0 = \hat{e}_z$$

Scalar products in spherical basis: $\mathbf{A} \cdot \mathbf{B} = -A_{+1}B_{-1} - A_{-1}B_{+1} + A_0 B_0$

Basic rotations:

$$R_{\mathbf{e_x}}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \quad R_{\mathbf{e_y}}(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \quad R_{\mathbf{e_z}}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Gaunt's formula

$$\int d\Omega\, Y^*_{lm}(\Omega) Y_{l_1 m_1}(\Omega) Y_{l_2 m_2}(\Omega) = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\, {}_u\langle l_1 l_2 m_1 m_2 | l_1 l_2 l m\rangle_c\, {}_u\langle l_1 l_2 0 0 | l_1 l_2 l 0\rangle_c.$$

Fermionic operators in the Fock space:

$$a_\nu | n_1 n_2 \dots 1_\nu \dots \rangle = (-1)^{\sum_{\mu=1}^{\nu-1} n_\mu} | n_1 n_2 \dots 0_\nu \dots \rangle$$

$$a^\dagger_\nu | n_1 n_2 \dots 0_\nu \dots \rangle = (-1)^{\sum_{\mu=1}^{\nu-1} n_\mu} | n_1 n_2 \dots 1_\nu \dots \rangle$$

$$\{a_\mu, a_\nu\} = 0 \qquad \{a^\dagger_\mu, a_\nu\} = \delta_{\mu\nu}$$

$$\hat{F} = \sum_{\mu,\nu} \langle \mu | \hat{f} | \nu \rangle a^\dagger_\mu a_\nu \qquad \hat{F} = \frac{1}{2} \sum_{\mu,\mu',\nu,\nu'} \langle \mu\mu' | \hat{g} | \nu\nu' \rangle a^\dagger_\mu a^\dagger_{\mu'} a_{\nu'} a_\nu$$

$$\psi(\mathbf{x},\sigma) = \sum_\mu \phi_\mu(\mathbf{x},\sigma) a_\mu \qquad \psi^\dagger(\mathbf{x},\sigma) = \sum_\mu \phi^*_\mu(\mathbf{x},\sigma) a^\dagger_\mu \qquad \hat{N} = \sum_\mu a^\dagger_\mu a_\mu$$

Relativistic theory:

metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) = g^{\mu\nu}$

scalar products $a \cdot b = a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu$

4-vectors: $x^\mu = (ct, \mathbf{x})$, $p^\mu = (E/c, \mathbf{p})$, $A^\mu = (\varphi/c, \mathbf{A})$

derivatives: $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla)$, and $\partial^\mu = \frac{\partial}{\partial x_\mu}$

Clifford algebra for the Dirac gamma-matrices: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4$

Dirac-Pauli representation:

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \vec{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

Taylor series expansions:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots \qquad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

Clebsch-Gordan coefficients, $3j$ symbols and some properties thereof:

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m}}{\sqrt{2j+1}} {}_u\langle j_1 j_2 m_1 m_2 | j_1 j_2 j \ -m\rangle_c$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_2 & m_3 & m_1 \end{pmatrix}$$

$$(-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$

$$= \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$$

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta_{j_3 j_3'} \delta_{m_3 m_3'}}{2j_3+1}$$

$$\sum_{j_3 m_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \quad \text{unless } m_1+m_2+m_3=0 \text{ and } \triangle(j_1,j_2,j_3)$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad \text{if } j_1+j_2+j_3 \text{ is odd}$$

$$\begin{pmatrix} j+\frac{1}{2} & j & \frac{1}{2} \\ m & -m-\frac{1}{2} & \frac{1}{2} \end{pmatrix} = (-1)^{j-m-\frac{1}{2}} \sqrt{\frac{j-m+\frac{1}{2}}{(2j+2)(2j+1)}}$$

$$\begin{pmatrix} j+1 & j & 1 \\ m & -m-1 & 1 \end{pmatrix} = (-1)^{j-m-1} \sqrt{\frac{(j-m)(j-m+1)}{(2j+3)(2j+2)(2j+1)}}$$

$$\begin{pmatrix} j+1 & j & 1 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m-1} \sqrt{\frac{(j+m+1)(j-m+1)}{(2j+3)(j+1)(2j+1)}}$$

$$\begin{pmatrix} j & j & 1 \\ m & -m-1 & 1 \end{pmatrix} = (-1)^{j-m} \sqrt{\frac{(j-m)(j+m+1)}{(j+1)(2j+1)(2j)}}$$

$$\begin{pmatrix} j & j & 1 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m} \frac{m}{\sqrt{(2j+1)(j+1)j}}$$

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m} \frac{1}{\sqrt{2j+1}}$$

$$\begin{pmatrix} j & j & 2 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m} \frac{3m^2 - j(j+1)}{\sqrt{(2j+3)(j+1)(2j+1)2j(2j-1)}}$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.