MA 1. Exercises.

1. For open $U \subseteq \mathbb{R}^n$, $u \in C_0^\infty(U)$ and for all times t define
$$E_U(u)(t) = \frac{1}{2}\int_U \left(|\nabla u(x,t)|^2 + \frac{1}{(c(x))^2}|u_t(x,t)|^2 \right) dx.$$
Show that $\frac{d}{dt} E(t) \equiv 0$.

2. Consider the system
$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ c^2(x)\Delta & 0 \end{pmatrix}}_{=P} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}.$$
Show that $P^* = -P$ (use pairs of C_0^∞-functions).

3. Verify that the following functions solve wave equation with sound speed c constant.

$\underline{n=1}$
$$u(x,t) = \frac{1}{2}\left(f(x + c\cancel{(x)}t) + f(x - ct) \right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$\underline{n=3}$
$$u(x,t) = \frac{t}{4\pi}\int_{S^2} g(x+ct\omega)\,d\omega + \frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2} f(x+ct\omega)\,d\omega \right)$$

Here $f = u\big|_{t=0}$ and $g = \partial_t u\big|_{t=0}$.

4. Prove that H_Ω is equivalent to $H_0^1(\Omega)$ when $g=0$.
H_Ω is the completion of $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ with the norm
$$\|(f,g)\|_{H_\Omega}^2 = \int_U \left(|\nabla f(x)|^2 + \frac{1}{c^2(x)}|g(x)|^2 \right) dx.$$

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5. Give details of the proof of following theorem:

<u>Thm</u>

(a) If for $T > T_0$ $\Lambda f = 0$, then $f = 0$

(b) For $T < T_0$ there is no uniqueness.

6. Suppose $f \in \mathcal{D}'(\Omega)$, $\mathrm{supp}(f)$ compact. Then

$f \in C^\infty(\Omega) \iff \forall N \; \exists C_N > 0 \; \forall |\xi| \geq 1 \quad |\hat{f}(\xi)| \leq C_N |\xi|^{-N}$.

$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$.

7. We have $WF(f) \subseteq \Omega \times (\mathbb{R}^n \setminus \{0\})$. Let $\pi : \Omega \times (\mathbb{R}^n \setminus \{0\}) \to \Omega$ be the projection.

Show that $\pi(WF(f)) \subseteq \mathrm{sing\,supp}(f)$

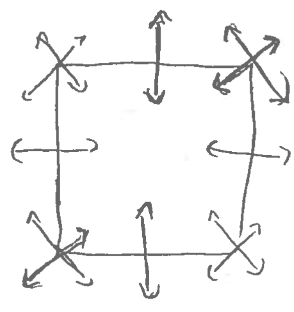
8. b) Suppose $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$.

a) $D \subseteq \mathbb{R}^n$ smooth bounded.

In both cases show $WF(\chi_D) = \{(x, \xi) \mid x \in \partial D,\ \xi \text{ normal to } \partial D\}$.

9. $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid \forall i \ |x_i| \leq 1\}$.

Show that $WF(\chi_D) = \{(x, \xi) \mid x \in \partial D, \xi \text{ as in picture}\}$



10. Consider the wave equation in $\mathbb{R}^1$ with sound speed $c \equiv 1$.

Compute $WF(u)$ in terms of $WF(f)$.

11. Suppose $\Omega = B(0, R) \subseteq \mathbb{R}^n$ and c is radial, i.e. $c(x) = c(|x|)$.

Show that c is non-trapping $\iff \dfrac{d}{dr}\left(\dfrac{r}{c(r)}\right) > 0$. $\quad (r = |x|)$.

12. $\Delta \phi = 0 \iff \phi$ minimises the energy $\int_\Omega |\nabla \phi(x)|^2 dx$.

13. Assume we have

(i) $u(x,t) = 0$ when $d(x,\Omega) > |T-t|$ and

(ii) $u(x,t) = 0$ when $d(x,\Omega) > t$,

where u solves the heat equation.

Show that for $-\frac{1}{2}T \le t \le \frac{3}{2}T$ we have $u(x,t) = 0$.

14. Suppose K is a compact operator and $\forall f$ $\|Kf\|^2_{H_\Omega} \le \|f\|^2_{H_\Omega}$

Prove that $\|K\|_{\mathcal{L}(H_\Omega)} < 1$.

15. Recover both $c(x)$ and f from $\Lambda_c f$ (photoacoustic experiment)