MA2, Exercise 1

Let (M,g) be a smooth compact manifold without boundary. Show that the following two ways to define $H^1(M)$ are equivalent.

a) Define $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, in the usual way via Fourier transform. Let $(U_j)_j$ be a finite cover of M and let $(\psi_j)_j$, $\psi_j \in C_0^\infty(U_j)$, be a partition of unity on M. Let $\alpha_j : U_j \to V_j \subset \mathbb{R}^n$ be homeomorphisms. Then for $u \in C^{-\infty}(M) = C^\infty(M)^*$ and $s \in \mathbb{R}$

$$u \in H^s(M) \iff (\psi_j u) \circ \alpha_j^{-1} \in H^s(V_j) \text{ for all } j.$$

b) H^1 is the completion of $C^\infty(M)$ in the norm

$$\|u\|^2_{H^1(M)} = \int_M \left(|u|^2 + |du|_g^2\right) d\mathrm{vol}_g .$$

MA2, Exercise 2

Let $U \subset \mathbb{R}^2$ and $\psi = (\psi_1, \psi_2) : U \to \mathbb{R}^2$. Show that ψ is conformal (i.e. $\psi_* g_e = e^{\rho} g_e$ for some $\rho \in C^\infty$) if and only if $\psi_1 + i\psi_2$ is holomorphic or antiholomorphic in $x_1 + i x_2$.

MA2, Exercise 3

we know that $\Delta_g Q_0 = Id + K$, where K has integral kernel $k \in C^\infty(\bar{M} \times \bar{M})$ and K is compact on $L^2(M)$. Show that there is a finite rank operator $L = \sum_{i=1}^{P} \langle \cdot, u_j \rangle v_j$ with $u_j, v_j \in C^\infty(M)$ and $v_j|_{\partial M} = 0$ such that $\Delta_g (Q_0 + L) = Id + \tilde{K}$, where $\tilde{K}$ is a compact operator on $L^2(M)$ with smooth integral kernel and $Id + \tilde{K}$ is invertible.

MA2, Exercise 4

Prove that

$$\int_M G_D(\xi,\xi')(\Delta_g \tilde{f})(\xi')\, d\mathrm{vol}_g(\xi')$$

$$= \tilde{f}(\xi) - \int_{\partial M} \partial_{n'} G_D(\xi,\xi')\big|_{\xi'=y} f(y)\, d\mathrm{vol}_{\partial M}(y).$$

Hint: write this integral as

$$\lim_{\varepsilon\to 0} \int_{M\setminus D(\xi,\varepsilon)} G_D(\xi,\xi')(\Delta_g \tilde{f})(\xi')\, d\mathrm{vol}_g(\xi')$$

and use Green's formula.

MA 2, Exercise 5

Prove that the Dirichlet to Neumann map N for $\mathbb{H}^2 = \{z \in \mathbb{C}; \operatorname{Im}(z) > 0\}$ is given by the Fourier multiplier

$$(Nf)(x) = \mathcal{F}^{-1}(|\xi|\,\hat{f}(\xi))(x).$$

Here $\Delta_{\mathbb{R}^2} u = 0$ on $\mathbb{H}^2$, $u|_{\partial\mathbb{H}^2} = f \in C_0^\infty(\mathbb{R})$ and

$$(Nf)(x) = -\partial_y u(x,y)|_{y=0}.$$

MA 2, Exercise 6

Using the Carleman estimate and the Riesz representation theorem prove that there is $C_0 > 0$ such that for all $f \in L^2(U)$ there is a unique solution u to

$$e^{-\varphi/h}(\Delta + V)e^{\varphi/h} u = f$$

and that it satisfies $\|u\|_{L^2} \leq C_0 \sqrt{h}\, \|f\|_{L^2}$.