The Calderón problem on Riemannian manifolds

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Preface

This text is an introduction to Calderón's inverse conductivity problem on Riemannian manifolds. This problem arises as a model for electrical imaging in anisotropic media, and it is one of the most basic inverse problems in a geometric setting. The problem is still largely open, but we will discuss recent developments based on complex geometrical optics and the geodesic X-ray transform in the case where one restricts to a fixed conformal class of conductivities.

This work is based on lectures for courses given at the University of Helsinki in 2010 and at Universidad Autónoma de Madrid in 2011. It has therefore the feeling of a set of lecture notes for a graduate course on the topic, together with exercises and also some problems which are open at the time of writing this. The main focus is on manifolds of dimension three and higher, where one has to rely on real variable methods instead of using complex analysis. The text can be considered as an introduction to geometric inverse problems, but also as an introduction to the use of real analysis methods in the setting of Riemannian manifolds.

Chapter 1 is an introduction to the Calderón problem on manifolds, stating the main questions studied in this text. Chapter 2 reviews basic facts on smooth and Riemannian manifolds, also discussing the Laplace-Beltrami operator and geodesics. Limiting Carleman weights, which turn out to exist on manifolds with a certain product structure, are treated in Chapter 3. Chapter 4 then proves Carleman estimates on manifolds with product structure. The proof uses a combination of the Fourier transform and eigenfunction expansions. Finally, in Chapter 5 we prove a uniqueness result for the inverse problem in certain geometries, based on inverting the geodesic X-ray transform.

As prerequisites for reading these notes, basic knowledge of real analysis, Riemannian geometry, and elliptic partial differential equations would be helpful. Familiarity with [**16**], [**12**, Chapters 1-5], and [**5**, Chapters 5-6] should be sufficient.

References. For a more thorough discussion on Calderón's inverse problem on manifolds and for references to known results, we refer to the introduction in [**4**]. General references for Chapter 2 include [**11**] for smooth manifolds, [**12**] for Riemannian manifolds, and [**20**] for the Laplace-Beltrami operator. Chapter 3 on limiting Carleman weights mostly follows [**4**, Section 2].

To motivate the definition of limiting Carleman weights, we use a little bit of semiclassical symbol calculus (for differential operators, not pseudodifferential ones). This is not covered in these notes, but on the other hand it is only used in Section 3.1 for motivation. See the lecture notes [**6**] for details on this topic (semiclassical calculus on manifolds is covered in an appendix).

The Fourier analysis proof of the Carleman estimates given in Chapter 4 is taken from [**10**]. Chapter 5, with the proof of the uniqueness result, follows [**4**, Sections 5 and 6]. For more details on the geodesic X-ray transform we refer the reader to [**18**] and [**4**, Section 7].

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CHAPTER 1

Introduction

To motivate the problems studied in this text, we start with the classical inverse conductivity problem of Calderón. This problem asks to determine the interior properties of a medium by making electrical measurements on its boundary.

In mathematical terms, one considers a bounded open set $\Omega \subseteq \mathbb{R}^n$ with smooth ($=C^\infty$) boundary, with electrical conductivity given by the matrix $\gamma(x) = (\gamma^{jk}(x))_{j,k=1}^n$. We assume that the functions γ^{jk} are smooth in $\overline{\Omega}$, and for each x the matrix $\gamma(x)$ is positive definite and symmetric. If $\gamma(x) = \sigma(x)I$ for some scalar function σ we say that the medium is *isotropic*, otherwise it is *anisotropic*. The electrical properties of anisotropic materials depend on direction. This is common in many applications such as in medical imaging (for instance cardiac muscle has a fiber structure and is an anisotropic conductor).

We seek to find the conductivity γ by prescribing different voltages on $\partial\Omega$ and by measuring the resulting current fluxes. If there are no sources or sinks of current in Ω, a boundary voltage f induces an electrical potential u which satisfies the conductivity equation

$$\left\{ \begin{aligned} \mathrm{div}(\gamma \nabla u) &= 0 && \text{in } \Omega, \\ u &= f && \text{on } \partial\Omega. \end{aligned} \right. \tag{1.1}$$

Since γ is positive definite this equation is elliptic and has a unique weak solution for any reasonable f (say in the L^2-based Sobolev space $H^{1/2}(\partial\Omega)$). The current flux on the boundary is given by the conormal derivative (where ν is the outer unit normal vector on $\partial\Omega$)

$$\Lambda_\gamma f := \gamma \nabla u \cdot \nu|_{\partial\Omega}.$$

The last expression is well defined also when γ is a matrix, and a suitable weak formulation shows that Λ_γ becomes a bounded map $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

The map Λ_γ is called the *Dirichlet-to-Neumann map*, DN map for short, since it maps the Dirichlet boundary value of a solution to what is essentially the Neumann boundary value. The DN map encodes the electrical boundary measurements (in the idealized case where we have infinite precision measurements for all possible data). The inverse problem is to find information about the conductivity matrix γ from the knowledge of the map Λ_γ.

The first important observation is that if γ is anisotropic, the full conductivity matrix can not be determined from Λ_γ. This is due to a transformation law for the conductivity equation under diffeomorphisms (that is, bijective maps F such that both F and F^{-1} are smooth up to the boundary).

Lemma. *If $F : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism and if $F|_{\partial\Omega} = \mathrm{Id}$, then*

$$\Lambda_{F_*\gamma} = \Lambda_\gamma.$$

Here $F_*\gamma$ is the pushforward of γ, defined by

$$F_*\gamma(\tilde{x}) = \left.\frac{(DF)\gamma(DF)^t}{|\det(DF)|}\right|_{F^{-1}(\tilde{x})}$$

where $DF = (\partial_k F_j)_{j,k=1}^n$ is the Jacobian matrix.

Exercise 1.1. Prove the lemma. (Hint: if u solves $\mathrm{div}(\gamma\nabla u) = 0$, show that $u \circ F^{-1}$ solves the analogous equation with conductivity $F_*\gamma$.)

The following conjecture for $n \geq 3$ is one of the most important open questions related to the inverse problem of Calderón. It has only been proved when $n = 2$.

Question 1.1. *(Anisotropic Calderón problem) Let γ_1, γ_2 be two smooth positive definite symmetric matrices in $\overline{\Omega}$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, show that $\gamma_2 = F_*\gamma_1$ for some diffeomorphism $F : \overline{\Omega} \to \overline{\Omega}$ with $F|_{\partial\Omega} = \mathrm{Id}$.*

In fact, the anisotropic Calderón problem is a question of geometric nature and can be formulated more generally on any Riemannian manifold. To do this, we replace the set $\overline{\Omega} \subseteq \mathbb{R}^n$ by a compact n-dimensional manifold M with smooth boundary ∂M, and the conductivity matrix γ by a smooth Riemannian metric g on M. On such a Riemannian manifold (M, g) there is a canonical second order elliptic operator Δ_g called the *Laplace-Beltrami operator*. In local coordinates

$$\Delta_g u = |g|^{-1/2} \frac{\partial}{\partial x_j} \left(|g|^{1/2} g^{jk} \frac{\partial u}{\partial x_k} \right).$$

We have written $g = (g_{jk})$ for the metric in local coordinates, $g^{-1} = (g^{jk})$ for its inverse matrix, and $|g|$ for $\det(g_{jk})$.

The Dirichlet problem for Δ_g analogous to (1.1) is

$$\left\{ \begin{array}{rl} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{array} \right. \tag{1.2}$$

The boundary measurements are given by the DN map

$$\Lambda_g f := \partial_\nu u|_{\partial M}$$

where $\partial_\nu u$ is the Riemannian normal derivative, given in local coordinates by $g^{jk}(\partial_{x_j} u)\nu_k$ where ν is the outer unit normal vector on ∂M. The inverse problem is to determine information on g from the DN map Λ_g.

There is a similar obstruction to uniqueness as for the conductivity equation, which is given by diffeomorphisms.

LEMMA. *If $F : M \to M$ is a diffeomorphism and if $F|_{\partial M} = \mathrm{Id}$, then*

$$\Lambda_{F^* g} = \Lambda_g.$$

Here $F^* g$ is the pullback of g, defined in local coordinates by

$$F^* g(x) = DF(x)^t g(F(x)) DF(x).$$

EXERCISE 1.2. Prove the lemma.

The geometric formulation of the anisotropic Calderón problem is as follows. We only state the question for $n \geq 3$, since again the two dimensional case is known (also the formulation for $n = 2$ would look slightly different since Δ_g has an additional conformal invariance then).

QUESTION 1.2. *(Anisotropic Calderón problem) Let (M, g_1) and (M, g_2) be two compact Riemannian manifolds of dimension $n \geq 3$ with smooth boundary, and assume that $\Lambda_{g_1} = \Lambda_{g_2}$. Show that $g_2 = F^* g_1$ for some diffeomorphism $F : M \to M$ with $F|_{\partial M} = \mathrm{Id}$.*

A function u satisfying $\Delta_g u = 0$ is called a harmonic function in (M, g). Note that if M is a subset of $\mathbb{R}^n$ with Euclidean metric, then this just gives the usual harmonic functions. Since $(u|_{\partial M}, \partial_\nu u|_{\partial M})$ is the Cauchy data of a function u, and since metrics satisfying $g_2 = F^* g_1$ are isometric in the sense of Riemannian geometry, the anisotropic Calderón problem reduces to the question: *Do the Cauchy data of all harmonic functions in (M, g) determine the manifold up to isometry?*

EXERCISE 1.3. Show that a positive answer to Question 1.2 would imply a positive answer to Question 1.1 when $n \geq 3$. (Hint: assume the boundary determination result that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $\det(\gamma_1^{jk}) = \det(\gamma_2^{jk})$ on $\partial\Omega$ [**13**].)

Instead of the full anisotropic Calderón problem, we will consider the simpler problem where the manifolds are assumed to be in the same conformal class. This means that the metrics g_1 and g_2 in M satisfy $g_2 = cg_1$ for some smooth positive function c on M. In this problem there is only one underlying metric g_1, and one is looking to determine a scalar function c. This covers the case of isotropic conductivities in Euclidean space, but if the metric is not Euclidean the problem still requires substantial geometric arguments.

The relevant question is as follows. It is known that any diffeomorphism $F : M \to M$ which satisfies $F|_{\partial M} = \mathrm{Id}$ and $F^* g_1 = cg_1$ must be the identity [**15**], so in this case there is no ambiguity arising from diffeomorphisms.

QUESTION 1.3. *(Anisotropic Calderón problem in a conformal class) Let (M, g_1) and (M, g_2) be two compact Riemannian manifolds of dimension $n \geq 3$ with smooth boundary which are in the same conformal class. If $\Lambda_{g_1} = \Lambda_{g_2}$, show that $g_1 = g_2$.*

EXERCISE 1.4. Using the fact on diffeomorphisms given above, show that a positive answer to Question 1.2 implies a positive answer to Question 1.3.

Finally, let us formulate one more question which will imply Question 1.3 but which is somewhat easier to study. This last question will be the one that the rest of these notes is devoted to.

The main point is the observation that the Laplace-Beltrami operator transforms under conformal scalings of the metric by

$$\Delta_{cg} u = c^{-\frac{n+2}{4}} (\Delta_g + q)(c^{\frac{n-2}{4}} u)$$

where $q = c^{\frac{n-2}{4}} \Delta_{cg}(c^{-\frac{n-2}{4}})$. It can be shown that for any smooth positive function c with $c|_{\partial M} = 1$ and $\partial_\nu c|_{\partial M} = 0$, one has

$$\Lambda_{cg} = \Lambda_{g,-q}$$

where $\Lambda_{g,V} : f \mapsto \partial_\nu u|_{\partial M}$ is the DN map for the Schrödinger equation

$$\left\{ \begin{array}{rl} (-\Delta_g + V)u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{array} \right. \tag{1.3}$$

For general V this last Dirichlet problem may not be uniquely solvable, but for $V = -q$ it is and the DN map is well defined since the Dirichlet problem for Δ_{cg} is uniquely solvable. We will make the standing assumption that all potentials V are such that (1.3) is uniquely solvable (this assumption could easily be removed by using Cauchy data sets). Then the last question is as follows. It is also of independent interest and a solution would have important consequences for the anisotropic Calderón problem, inverse problems for Maxwell equations, and inverse scattering theory.

QUESTION 1.4. *Let (M, g) be a compact Riemannian manifold with smooth boundary, and let V_1 and V_2 be two smooth functions on M. If $\Lambda_{g,V_1} = \Lambda_{g,V_2}$, show that $V_1 = V_2$.*

EXERCISE 1.5. Prove the above identities for Δ_{cg} and Λ_{cg}. Show that a positive answer to Question 1.4 implies a positive answer to Question 1.3. (You may assume the boundary determination result that $\Lambda_{cg} = \Lambda_g$ implies $c|_{\partial M} = 1$ and $\partial_\nu c|_{\partial M} = 0$ [**13**].)

CHAPTER 2

Riemannian geometry

2.1. Smooth manifolds

Manifolds. We recall some basic definitions from the theory of smooth manifolds. We will consistently also consider manifolds with boundary.

Definition. A *smooth n-dimensional manifold* is a second countable Hausdorff topological space together with an open cover $\{U_\alpha\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to \tilde{U}_\alpha$ such that each $\tilde{U}_\alpha$ is an open set in $\mathbb{R}^n$, and $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map whenever $U_\alpha \cap U_\beta$ is nonempty.

Any family $\{(U_\alpha, \varphi_\alpha)\}$ as above is called an *atlas*. Any atlas gives rise to a maximal atlas, called a *smooth structure*, which is not strictly contained in any other atlas. We assume that we are always dealing with the maximal atlas. The pairs $(U_\alpha, \varphi_\alpha)$ are called *charts*, and the maps φ_α are called *local coordinate systems* (one usually writes $x = \varphi_\alpha$ and thus identifies points $p \in U_\alpha$ with points $x(p) \in \tilde{U}_\alpha$ in $\mathbb{R}^n$).

Definition. A *smooth n-dimensional manifold with boundary* is a second countable Hausdorff topological space together with an open cover $\{U_\alpha\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to \tilde{U}_\alpha$ such that each $\tilde{U}_\alpha$ is an open set in $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \,;\, x_n \geq 0\}$, and $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map whenever $U_\alpha \cap U_\beta$ is nonempty.

Here, if $A \subseteq \mathbb{R}^n$ we say that a map $F : A \to \mathbb{R}^n$ is smooth if it extends to a smooth map $\tilde{A} \to \mathbb{R}^n$ where $\tilde{A}$ is an open set in $\mathbb{R}^n$ containing A.

If M is a manifold with boundary we say that p is a boundary point if $\varphi(p) \in \partial \mathbb{R}^n_+$ for some chart φ, and an interior point if $\varphi(p) \in \mathrm{int}(\mathbb{R}^n_+)$ for some φ. We write ∂M for the set of boundary points and M^{int} for the set of interior points. Since M is not assumed to be embedded in any larger space, these definitions may differ from the usual ones in point set topology.

Exercise 2.1. If M is a manifold with boundary, show that the sets M^{int} and ∂M are always disjoint.

To clarify the relations between the definitions, note that a manifold is always a manifold with boundary (the boundary being empty), but a manifold with boundary is a manifold iff the boundary is empty (by the above exercise). However, we will loosely refer to manifolds both with and without boundary as 'manifolds'.

We have the following classes of manifolds:

- A *closed manifold* is compact, connected, and has no boundary
  - Examples: the sphere S^n, the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$
- An *open manifold* has no boundary and no component is compact
  - Examples: open subsets of $\mathbb{R}^n$, strict open subsets of a closed manifold
- A *compact manifold with boundary* is a manifold with boundary which is compact as a topological space
  - Examples: the closures of bounded open sets in $\mathbb{R}^n$ with smooth boundary, the closures of open sets with smooth boundary in closed manifolds

Smooth maps.

Definition. Let $f : M \to N$ be a map between two manifolds. We say that f is *smooth* near a point p if $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth for some charts (U, φ) of M and (V, ψ) of N such that $p \in U$ and $f(U) \subseteq V$. We say that f is *smooth* in a set $A \subseteq M$ if it is smooth near any point of A. The set of all maps $f : M \to N$ which are smooth in A is denoted by $C^\infty(A, N)$. If $N = \mathbb{R}$ we write $C^\infty(A, N) = C^\infty(A)$.

Summation convention. Below and throughout these notes we will apply the *Einstein summation convention*: repeated indices in lower and upper position are summed. For instance, the expression

$$a_{jkl} b^j c^k$$

is shorthand for

$$\sum_{j,k} a_{jkl} b^j c^k.$$

The summation indices run typically from 1 to n, where n is the dimension of the manifold.

Tangent bundle.

Definition. Let $p \in M$. A *derivation* at p is a linear map $v : C^\infty(M) \to \mathbb{R}$ which satisfies $v(fg) = (vf)g(p) + f(p)(vg)$. The *tangent space* T_pM is the vector space consisting of all derivations at p. Its elements are called *tangent vectors*.

The tangent space T_pM is an n-dimensional vector space when $\dim(M) = n$. If x is a local coordinate system in a neighborhood U of p, the *coordinate vector fields* ∂_j are defined for any $q \in U$ to be the derivations

$$\partial_j|_q f := \frac{\partial}{\partial x_j}(f \circ x^{-1})(x(q)), \quad j = 1, \ldots, n.$$

Then $\{\partial_j|_q\}$ is a basis of T_qM, and any $v \in T_qM$ may be written as $v = v^j \partial_j$.

The *tangent bundle* is the disjoint union

$$TM := \bigvee_{p \in M} T_pM.$$

The tangent bundle has the structure of a $2n$-dimensional manifold defined as follows. For any chart (U, x) of M we represent elements of T_qM for $q \in U$ as $v = v^j(q)\partial_j|_q$, and define a map $\tilde{\varphi} : TU \to \mathbb{R}^{2n}, \tilde{\varphi}(q, v) = (x(q), v^1(q), \ldots, v^n(q))$. The charts $(TU, \tilde{\varphi})$ are called the *standard charts* of TM and they define a smooth structure on TM.

Exercise 2.2. Prove that T_pM is an n-dimensional vector space spanned by $\{\partial_j\}$ also when M is a manifold with boundary.

Cotangent bundle. The dual space of a vector space V is

$$V^* := \{u : V \to \mathbb{R}\,;\, u \text{ linear}\}.$$

The dual space of T_pM is denoted by T_p^*M and is called the *cotangent space* of M at p. Let x be local coordinates in U, and let ∂_j be the coordinate vector fields that span T_qM for $q \in U$. We denote by dx^j the elements of the dual basis of T_q^*M, so that any $\xi \in T_q^*M$ can be written as $\xi = \xi_j\, dx^j$. The dual basis is characterized by

$$dx^j(\partial_k) = \delta_{jk}.$$

The *cotangent bundle* is the disjoint union

$$T^*M = \bigvee_{p \in M} T_p^* M.$$

This becomes a $2n$-dimensional manifold by defining for any chart (U, φ) of M a chart $(T^*U, \tilde{\varphi})$ of T^*M by $\tilde{\varphi}(q, \xi_j \, dx^j) = (\varphi(q), \xi_1, \ldots, \xi_n)$.

Tensor bundles. If V is a finite dimensional vector space, the space of (covariant) k-tensors on V is

$$T^k(V) := \{u : \underbrace{V \times \ldots \times V}_{k \text{ copies}} \to \mathbb{R}\,;\, u \text{ linear in each variable}\}.$$

The *k-tensor bundle* on M is the disjoint union

$$T^k M = \bigvee_{p \in M} T^k(T_p M).$$

If x are local coordinates in U and dx^j is the basis for $T_q^* M$, then each $u \in T^k(T_q M)$ for $q \in U$ can be written as

$$u = u_{j_1 \cdots j_k} dx^{j_1} \otimes \ldots \otimes dx^{j_k}$$

Here $\otimes$ is the *tensor product*

$$T^k(V) \times T^{k'}(V) \to T^{k+k'}(V), \quad (u, u') \mapsto u \otimes u',$$

where for $v \in V^k, v' \in V^{k'}$ we have

$$(u \otimes u')(v, v') := u(v) u'(v').$$

It follows that the elements $dx^{j_1} \otimes \ldots \otimes dx^{j_k}$ span $T^k(T_q M)$. Similarly as above, $T^k M$ has the structure of a smooth manifold (of dimension $n + n^k$).

Exterior powers. The space of alternating k-tensors is

$$A^k(V) := \{u \in T^k(V)\,;\, u(v_1, \ldots, v_k) = 0 \text{ if } v_i = v_j \text{ for some } i \neq j\}.$$

This gives rise to the bundle

$$\Lambda^k(M) := \bigvee_{p \in M} A^k(T_p M).$$

To describe a basis for $A^k(T_p M)$, we introduce the *wedge product*

$$A^k(V) \times A^{k'}(V) \to A^{k+k'}(V), \ (\omega, \omega') \mapsto \omega \wedge \omega' := \frac{(k+k')!}{k!(k')!} \mathrm{Alt}(\omega \otimes \omega'),$$

where $\mathrm{Alt} : T^k(V) \to A^k(V)$ is the projection to alternating tensors,

$$\mathrm{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

We have written S_k for the group of permutations of $\{1, \ldots, k\}$, and $\mathrm{sgn}(\sigma)$ for the signature of $\sigma \in S_k$.

If x is a local coordinate system in U, then a basis of $A^k(T_pM)$ is given by

$$\{dx^{j_1} \wedge \ldots dx^{j_k}\}_{1 \leq j_1 < j_2 < \ldots < j_k \leq n}.$$

Again, $\Lambda^k(M)$ is a smooth manifold (of dimension $n + \binom{n}{k}$).

Exercise 2.3. Show that Alt maps $T^k(V)$ into $A^k(V)$ and that $(\mathrm{Alt})^2 = \mathrm{Alt}$.

Smooth sections. The above constructions of the tangent bundle, cotangent bundle, tensor bundles, and exterior powers are all examples of *vector bundles* with base manifold M. We will not need a precise definition here, but just note that in each case there is a natural vector space over any point $p \in M$ (called the *fiber over* p). A *smooth section* of a vector bundle E over M is a smooth map $s : M \to E$ such that for each $p \in M$, $s(p)$ belongs to the fiber over p. The space of smooth sections of E is denoted by $C^\infty(M, E)$.

We have the following terminology:

- $C^\infty(M, TM)$ is the set of *vector fields* on M,
- $C^\infty(M, T^*M)$ is the set of 1-*forms* on M,
- $C^\infty(M, T^kM)$ is the set of k-*tensor fields* on M,
- $C^\infty(M, \Lambda^kM)$ is the set of *(differential)* k-*forms* on M.

Let x be local coordinates in a set U, and let ∂_j and dx^j be the coordinate vector fields and 1-forms in U which span T_qM and T_q^*M, respectively, for $q \in U$. In these local coordinates,

- a vector field X has the expression $X = X^j \partial_j$,
- a 1-form α has expression $\alpha = \alpha_j \, dx^j$,
- a k-tensor field u can be written as

$$u = u_{j_1 \cdots j_k} dx^{j_1} \otimes \ldots \otimes dx^{j_k},$$

- a k-form ω has the form

$$\omega = \omega_I \, dx^I$$

where $I = (i_1, \ldots, i_k)$ and $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, with the sum being over all I such that $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Here, the component functions $X^j, \alpha_j, u_{j_1 \cdots j_k}, \omega_I$ are all smooth real valued functions in U.

Note that a vector field $X \in C^\infty(M, TM)$ gives rise to a linear map $X : C^\infty(M) \to C^\infty(M)$ via $Xf(p) = X(p)f$.

Example. Some examples of the smooth sections that will be encountered in this text are:

- Vector fields: the gradient vector field $\mathrm{grad}(f)$ for $f \in C^\infty(M)$, coordinate vector fields ∂_j in a chart U
- One-forms: the exterior derivative df for $f \in C^\infty(M)$
- 2-tensor fields: Riemannian metrics g, Hessians $\mathrm{Hess}(f)$ for $f \in C^\infty(M)$
- k-forms: the volume form dV in Riemannian manifold (M, g), the volume form dS of the boundary ∂M

Changes of coordinates. We consider the transformation law for k-tensor fields under changes of coordinates, or more generally under pullbacks by smooth maps. If $F : M \to N$ is a smooth map, the *pullback* by F is the map $F^* : C^\infty(N, T^k N) \to C^\infty(M, T^k M)$,

$$(F^* u)_p(v_1, \ldots, v_k) = u_{F(p)}(F_* v_1, \ldots, F_* v_k)$$

where $v_1, \ldots, v_k \in T_p\tilde{M}$. Here $F_* : T_p M \to T_{F(p)} N$ is the pushforward, defined by $(F_* v)f = v(f \circ F)$ for $v \in T_p M$ and $f \in C^\infty(N)$. Clearly F^* pulls back k-forms on N to k-forms on M.

The pullback satisfies

- $F^*(fu) = (f \circ F) F^* u$
- $F^*(u \otimes u') = F^* u \otimes F^* u'$
- $F^*(\omega \wedge \omega') = F^* \omega \wedge F^* \omega'$

In terms of local coordinates, the pullback acts by

- $F^* f = f \circ F$ if f is a smooth function (=0-form)
- $F^*(\alpha_j \, dx^j) = (\alpha_j \circ F) \, d(x^j \circ F)$ if α is a 1-form

and it has similar expressions for higher order tensors.

Exterior derivative. The exterior derivative d is a first order differential operator mapping differential k-forms to $k+1$-forms. It can be defined first on 0-forms (that is, smooth functions f) by the local coordinate expression

$$df := \frac{\partial f}{\partial x_j} dx^j.$$

In general, if $\omega = \omega_I \, dx^I$ is a k-form we define

$$d\omega := d\omega_I \wedge dx^I.$$

It turns out that this definition is independent of the choice of coordinates, and one obtains a linear map $d : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^{k+1})$. It has the properties

- $d^2 = 0$
- $d = 0$ on n-forms
- $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^k \omega \wedge d\omega'$ for a k-form ω, k'-form ω'
- $F^* d\omega = dF^* \omega$

Exercise 2.4. If f is a smooth function and $V = (V_1, V_2, V_3)$ is a smooth vector field on $\mathbb{R}^3$, show that the exterior derivative is related to the gradient, curl, and divergence by

$$\begin{aligned} df &= (\nabla f)_j \, dx^j, \\ d(V_j \, dx^j) &= (\nabla \times V)_j \, dx^{\widehat{j}}, \\ d(V_j \, dx^{\widehat{j}}) &= (\nabla \cdot V) \, dx^1 \wedge dx^2 \wedge dx^3, \\ d(f \, dx^1 \wedge dx^2 \wedge dx^3) &= 0. \end{aligned}$$

Here $dx^{\widehat{1}} := dx^2 \wedge dx^3,\ dx^{\widehat{2}} := dx^3 \wedge dx^1,\ dx^{\widehat{3}} := dx^1 \wedge dx^2$.

Partition of unity. A major reason for including the condition of second countability in the definition of manifolds is to ensure the existence of partitions of unity. These make it possible to make constructions in local coordinates and then glue them together to obtain a global construction.

Theorem 2.1. *Let M be a manifold and let $\{U_\alpha\}$ be an open cover. There exists a family of C^∞ functions $\{\chi_\alpha\}$ on M such that $0 \leq \chi_\alpha \leq 1$,* $\mathrm{supp}(\chi_\alpha) \subseteq U_\alpha$*, any point of M has a neighborhood which intersects only finitely many of the sets* $\mathrm{supp}(\chi_\alpha)$*, and further*

$$\sum_\alpha \chi_\alpha = 1 \quad \text{in } M.$$

Integration on manifolds. The natural objects that can be integrated on an n-dimensional manifold are the differential n-forms. This is due to the transformation law for n-forms in $\mathbb{R}^n$ under smooth diffeomorphisms F in $\mathbb{R}^n$,

$$F^*(dx^1 \wedge \cdots \wedge dx^n) = (\det DF) dx^1 \wedge \cdots \wedge dx^n.$$

This is almost the same as the transformation law for integrals in $\mathbb{R}^n$ under changes of variables, the only difference being that in the latter the factor $|\det DF|$ instead $\det DF$ appears. To define an invariant integral, we therefore need to make sure that all changes of coordinates have positive Jacobian.

Definition. If M admits a smooth nonvanishing n-form we say that M is *orientable.* An *oriented manifold* is a manifold together with a given nonvanishing n-form.

If M is oriented with a given n-form Ω, a basis $\{v_1, \dots, v_n\}$ of T_pM is called *positive* if $\Omega(v_1, \dots, v_n) > 0$. There are many n-forms on an oriented manifold which give the same positive bases; we call any such n-form an *orientation form.* If (U, φ) is a connected coordinate chart, we say that this chart is *positive* if the coordinate vector fields $\{\partial_1, \dots, \partial_n\}$ form a positive basis of T_qM for all $q \in M$.

A map $F : M \to N$ between two oriented manifolds is said to be *orientation preserving* if it pulls back an orientation form on N to an orientation form of M. In terms of local coordinates given by positive charts, one can see that a map is orientation preserving iff its Jacobian determinant is positive.

Example. The standard orientation of $\mathbb{R}^n$ is given by the n-form $dx^1 \wedge \cdots \wedge dx^n$, where x are the Cartesian coordinates.

If ω is a compactly supported n-form in $\mathbb{R}^n$, we may write $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$ for some smooth compactly supported function f. Then the integral of ω is defined by

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f(x)\, dx^1 \cdots dx^n.$$

If ω is a smooth 1-form in a manifold M whose support is compactly contained in U for some positive chart (U, φ), then the integral of ω

over M is defined by

$$\int_M \omega := \int_{\varphi(U)} ((\varphi)^{-1})^* \omega.$$

Finally, if ω is a compactly supported n-form in a manifold M, the integral of ω over M is defined by

$$\int_M \omega := \sum_j \int_{U_j} \chi_j \omega.$$

where $\{U_j\}$ is some open cover of $\operatorname{supp}(\omega)$ by positive charts, and $\{\chi_j\}$ is a partition of unity subordinate to the cover $\{U_j\}$.

Exercise 2.5. Prove that the definition of the integral is independent of the choice of positive charts and the partition of unity.

The following result is a basic integration by parts formula which implies the usual theorems of Gauss and Green.

Theorem 2.2. *(Stokes theorem) If M is an oriented manifold with boundary and if ω is a compactly supported $(n-1)$-form on M, then*

$$\int_M d\omega = \int_{\partial M} i^* \omega$$

where $i : \partial M \to M$ is the natural inclusion.

Here, if M is an oriented manifold with boundary, then ∂M has a natural orientation defined as follows: for any point $p \in \partial M$, a basis $\{E_1, \ldots, E_{n-1}\}$ of $T_p(\partial M)$ is defined to be positive if $\{N_p, E_1, \ldots, E_{n-1}\}$ is a positive basis of T_pM where N is some outward pointing vector field near ∂M (that is, there is a smooth curve $\gamma : [0, \varepsilon) \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = -N_p$).

Exercise 2.6. Prove that any manifold with boundary has an outward pointing vector field, and show that the above definition gives a valid orientation on ∂M.

2.2. Riemannian manifolds

Riemannian metrics. If u is a 2-tensor field on M, we say that u is *symmetric* if $u(v,w) = u(w,v)$ for any tangent vectors v, w, and that u is *positive definite* if $u(v,v) > 0$ unless $v = 0$.

DEFINITION. Let M be a manifold. A *Riemannian metric* is a symmetric positive definite 2-tensor field g on M. The pair (M,g) is called a *Riemannian manifold*.

If g is a Riemannian metric on M, then $g_p : T_pM \times T_pM$ is an inner product on T_pM for any $p \in M$. We will write

$$\langle v, w\rangle := g(v,w), \qquad |v| := \langle v, v\rangle^{1/2}.$$

In local coordinates, a Riemannian metric is just a positive definite symmetric matrix. To see this, let (U, x) be a chart of M, and write $v, w \in T_qM$ for $q \in U$ in terms of the coordinate vector fields ∂_j as $v = v^j \partial_j$, $w = w^j \partial_j$. Then

$$g(v,w) = g(\partial_j, \partial_k) v^j w^k.$$

This shows that g has the local coordinate expression

$$g = g_{jk} dx^j \otimes dx^k$$

where $g_{jk} := g(\partial_j, \partial_k)$ and the matrix $(g_{jk})_{j,k=1}^n$ is symmetric and positive definite. We will also write $(g^{jk})_{j,k=1}^n$ for the inverse matrix of (g_{jk}), and $|g| := \det(g_{jk})$ for the determinant.

EXAMPLE. Some examples of Riemannian manifolds:

1. (Euclidean space) If Ω is a bounded open set in $\mathbb{R}^n$, then (Ω, e) is a Riemannian manifold if e is the *Euclidean metric* for which $e(v,w) = v \cdot w$ is the Euclidean inner product of $v, w \in T_p\Omega \approx \mathbb{R}^n$. In Cartesian coordinates, e is just the identity matrix.
2. If Ω is as above, then more generally (Ω, g) is a Riemannian manifold if $g(x) = (g_{jk}(x))_{j,k=1}^n$ is any family of positive definite symmetric matrices whose elements depend smoothly on $x \in \Omega$.
3. If Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary, then $(\overline{\Omega}, g)$ is a compact Riemannian manifold with boundary if $g(x)$ is a family of positive definite symmetric matrices depending smoothly on $x \in \overline{\Omega}$.

4. (Hypersurfaces) Let S be a smooth hypersurface in $\mathbb{R}^n$ such that $S = f^{-1}(0)$ for some smooth function $f : \mathbb{R}^n \to \mathbb{R}$ which satisfies $\nabla f \neq 0$ when $f = 0$. Then S is a smooth manifold of dimension $n-1$, and the tangent space T_pS for any $p \in S$ can be identified with $\{v \in \mathbb{R}^n\,;\, v \cdot \nabla f(p) = 0\}$. Using this identification, we define an inner product $g_p(v, w)$ on T_pS by taking the Euclidean inner product of v and w interpreted as vectors in $\mathbb{R}^n$. Then (S, g) is a Riemannian manifold, and g is called the *induced Riemannian metric* on S (this metric being induced by the Euclidean metric in $\mathbb{R}^n$).
5. (Model spaces) The model spaces of Riemannian geometry are the Euclidean space $(\mathbb{R}^n, e)$, the sphere (S^n, g) where S^n is the unit sphere in $\mathbb{R}^{n+1}$ and g is the induced Riemannian metric, and the hyperbolic space (H^n, g) which may be realized by taking H^n to be the unit ball in $\mathbb{R}^n$ with metric $g_{jk}(x) = \frac{4}{(1-|x|^2)^2}\delta_{jk}$.

The Riemannian metric allows to measure lengths and angles of tangent vectors on a manifold, the *length* of a vector $v \in T_pM$ being $|v|$ and the *angle* between two vectors $v, w \in T_pM$ being the number $\theta(v, w) \in [0, \pi]$ which satisfies

$$\cos\theta(v, w) := \frac{\langle v, w\rangle}{|v||w|}. \tag{2.1}$$

Physically, one may think of a Riemannian metric g as the resistivity of a conducting medium (in the introduction, the conductivity matrix (γ^{jk}) corresponded formally to $(|g|^{1/2}g^{jk})$), or as the inverse of sound speed squared in a medium where acoustic waves propagate (if a medium $\Omega \subseteq \mathbb{R}^n$ has scalar sound speed $c(x)$ then a natural Riemannian metric is $g_{jk}(x) = c(x)^{-2}\delta_{jk}$). In the latter case, regions where g is large (resp. small) correspond to low velocity regions (resp. high velocity regions). We will later define geodesics, which are length minimizing curves on a Riemannian manifold, and these tend to avoid low velocity regions as one would expect.

Exercise 2.7. Use a partition of unity to prove that any smooth manifold M admits a Riemannian metric.

Raising and lowering of indices. On a Riemannian manifold (M, g) there is a canonical way of converting tangent vectors into cotangent vectors and vice versa. We define a map

$$T_pM \to T_p^*M, \quad v \mapsto v^\flat$$

by requiring that $v^\flat(w) = \langle v, w\rangle$. This map (called the 'flat' operator) is an isomorphism, which is given in local coordinates by

$$(v^j \partial_j)^\flat = v_j \, dx^j, \quad \text{where } v_j := g_{jk} v^k.$$

We say that $v^\flat$ is the cotangent vector obtained from v by *lowering indices.* The inverse of this map is the 'sharp' operator

$$T_p^* M \to T_p M, \quad \xi \mapsto \xi^\sharp$$

given in local coordinates by

$$(\xi_j \, dx^j)^\sharp = \xi^j \partial_j, \quad \text{where } \xi^j := g^{jk} \xi_k.$$

We say that $\xi^\sharp$ is obtained from ξ by *raising indices* with respect to the metric g.

A standard example of this construction is the *metric gradient.* If $f \in C^\infty(M)$, the metric gradient of f is the vector field

$$\mathrm{grad}(f) := (df)^\sharp.$$

In local coordinates, $\mathrm{grad}(f) = g^{jk}(\partial_j f)\partial_k$.

Inner products of tensors. If (M, g) is a Riemannian manifold, we can use the Riemannian metric g to define inner products of differential forms and other tensors in a canonical way. We will mostly use the inner product of 1-forms, defined via the sharp operator by

$$\langle \alpha, \beta \rangle := \langle \alpha^\sharp, \beta^\sharp \rangle.$$

In local coordinates one has $\langle \alpha, \beta \rangle = g^{jk} \alpha_j \beta_k$ and $g^{jk} = \langle dx^j, dx^k \rangle$.

More generally, if u and v are k-tensor fields with local coordinate representations $u = u_{i_1 \cdots i_k} \, dx^{i_1} \otimes \cdots \otimes dx^{i_k}$, $v = v_{i_1 \cdots i_k} \, dx^{i_1} \otimes \cdots \otimes dx^{i_k}$, we define

$$\langle u, v \rangle := g^{i_1 j_1} \cdots g^{i_k j_k} u_{i_1 \cdots i_k} v_{j_1 \cdots j_k}. \tag{2.2}$$

This definition turns out to be independent of the choice of coordinates, and it gives a valid inner product on k-tensor fields.

Orthonormal frames. If U is an open subset of M, we say that a set $\{E_1, \ldots, E_n\}$ of vector fields in U is a *local orthonormal frame* if $\{E_1(q), \ldots, E_n(q)\}$ forms an orthonormal basis of $T_q M$ for any $q \in U$.

LEMMA 2.3. *(Local orthonormal frame) If (M, g) is a Riemannian manifold, then for any point $p \in M$ there is a local orthonormal frame in some neighborhood of p.*

If $\{E_j\}$ is a local orthonormal frame, the dual frame $\{\varepsilon^j\}$ which is characterized by $\varepsilon^j(E_k) = \delta_{jk}$ gives an orthonormal basis of T_q^*M for any q near p. The inner product in (2.2) is the unique inner product on k-tensor fields such that $\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}\}$ gives an orthonormal basis of $T^k(T_qM)$ for q near p whenever $\{\varepsilon^j\}$ is a local orthonormal frame of 1-forms near p.

Exercise 2.8. Prove the lemma by applying the Gram-Schmidt orthonormalization procedure to a basis $\{\partial_j\}$ of coordinate vector fields, and prove the statements after the lemma.

Volume form, integration, and L^2 Sobolev spaces. From this point on, *all Riemannian manifolds will be assumed to be oriented.* Clearly near any point p in (M, g) there is a positive local orthonormal frame (that is, a local orthonormal frame $\{E_j\}$ which gives a positive orthonormal basis of T_qM for q near p).

Lemma 2.4. *(Volume form) Let (M, g) be a Riemannian manifold. There is a unique n-form on M, denoted by dV and called the* volume form, *such that $dV(E_1, \ldots, E_n) = 1$ for any positive local orthonormal frame $\{E_j\}$. In local coordinates*

$$dV = |g|^{1/2}\, dx^1 \wedge \ldots \wedge dx^n.$$

Exercise 2.9. Prove this lemma.

If f is a function on (M, g), we can use the volume form to obtain an n-form $f\, dV$. The integral of f over M is then defined to be the integral of the n-form $f\, dV$. Thus, on a Riemannian manifold there is a canonical way to integrate functions (instead of just n-forms).

If $u, v \in C^\infty(M)$ are complex valued functions, we define the L^2 inner product by

$$(u, v) = (u, v)_{L^2(M)} := \int_M u\bar{v}\, dV.$$

The completion of $C^\infty(M)$ with respect to this inner product is a Hilbert space denoted by $L^2(M)$ or $L^2(M, dV)$. It consists of square integrable functions defined almost everywhere on M with respect to the measure dV. The L^2 norm is defined by

$$\|u\| = \|u\|_{L^2(M)} := (u, u)_{L^2(M)}^{1/2}.$$

Similarly, we may define the spaces of square integrable k-forms or k-tensor fields, denoted by $L^2(M, \Lambda^k M)$ or $L^2(M, T^k M)$, by using the inner product

$$(u, v) := \int_M \langle u, \bar{v} \rangle \, dV, \quad u, v \in C^\infty(M, T^k M) \text{ complex valued.}$$

We may use the above inner products to give a definition of low order Sobolev spaces on Riemannian manifolds which does not involve local coordinates. We define the $H^1(M)$ inner product

$$(u, v)_{H^1(M)} := (u, v) + (du, dv), \quad u, v \in C^\infty(M) \text{ complex valued.}$$

The space $H^1(M)$ (resp. $H^1_0(M)$) is defined to be the completion of $C^\infty(M)$ (resp. $C^\infty_c(M^{\mathrm{int}})$) with respect to this inner product. These are subspaces of $L^2(M)$ which have first order weak derivatives in $L^2(M)$, and they coincide with the spaces defined in the usual way by using local coordinates. Also, we define $H^{-1}(M)$ to be the dual space of $H^1_0(M)$.

Codifferential. Using the inner product on k-forms, we can define the codifferential operator δ as the adjoint of the exterior derivative via the relation

$$(\delta u, v) = (u, dv)$$

where $u \in C^\infty(M, \Lambda^k)$ and $v \in C^\infty_c(M^{\mathrm{int}}, \Lambda^{k-1})$. It can be shown that δ gives a well-defined map

$$\delta : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^{k-1}).$$

We will only use δ for 1-forms, and in this case the operator can be easily defined by a local coordinate expression. Let α be a 1-form in M, let (U, x) be a chart and let $\varphi \in C^\infty_c(U)$. One computes in local coordinates

$$\begin{aligned}(\alpha, dv) &= \int_U \langle \alpha, d\bar{v} \rangle \, dV = \int_U g^{jk} \alpha_j \overline{\partial_k v} \, |g|^{1/2} \, dx \\ &= -\int_U |g|^{-1/2} \partial_k (|g|^{1/2} g^{jk} \alpha_j) \bar{v} \, dV.\end{aligned}$$

This computation shows that the function $\delta\alpha$, defined in local coordinates by

$$\delta\alpha := -|g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \alpha_k),$$

is a smooth function in M and satisfies $(\delta\alpha, v) = (\alpha, dv)$.

It follows that $\delta\alpha$ is related to the divergence of vector fields by $\delta\alpha = -\mathrm{div}(\alpha^\sharp)$, where the divergence is defined by

$$\mathrm{div}(X) := |g|^{-1/2}\partial_j(|g|^{1/2}X^j).$$

EXERCISE 2.10. (Hodge star operator) Let (M, g) be a Riemannian manifold of dimension n. If ω and η are k-forms on M, show that the identity

$$\omega \wedge *\eta = \langle \omega, \eta \rangle \, dV$$

determines uniquely a linear operator (called the *Hodge star operator*)

$$* : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^{n-k}).$$

Prove the following properties:

- $** = (-1)^{k(n-k)}$ on k-forms
- $*1 = dV$
- $*(\varepsilon^1 \wedge \ldots \wedge \varepsilon^k) = \varepsilon^{k+1} \wedge \ldots \wedge \varepsilon^n$ whenever $(\varepsilon^1, \ldots, \varepsilon^n)$ is a positive local orthonormal frame on T^*M
- $\langle *\omega, \eta \rangle = -\langle \omega, *\eta \rangle$ when ω, η are 1-forms and $\dim(M) = 2$ (that is, on 2D manifolds the Hodge star on 1-forms corresponds to rotation by 90°)

Prove that the operator

$$\delta := (-1)^{(k-1)(n-k)-1} * d * \quad \text{on } k\text{-forms}$$

gives a map $\delta : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^{k-1})$ satisfying $(\delta u, v) = (u, dv)$ for compactly supported v, and thus gives a valid definition of the codifferential on forms of any order.

Conformality. As the last topic in this section, we discuss the notion of conformality of manifolds.

DEFINITION. Two metrics g_1 and g_2 on a manifold M are called *conformal* if $g_2 = cg_1$ for a smooth positive function c on M. A diffeomorphism $f : (M, g) \to (M', g')$ is called a *conformal transformation* if f^*g' is conformal to g, that is,

$$f^*g' = cg.$$

Two Riemannian manifolds are called conformal if there is a conformal transformation between them.

We relate this definition of conformality to the standard one in complex analysis via the concept of angle $\theta(v,w) = \theta_g(v,w) \in [0,\pi]$ defined in (2.1).

LEMMA 2.5. *(Conformal = angle-preserving) Let $f : (M,g) \to (M',g')$ be a diffeomorphism. The following are equivalent.*

(1) *f is a conformal transformation.*
(2) *f preserves angles in the sense that $\theta_g(v,w) = \theta_{g'}(f_*v, f_*w)$.*

EXERCISE 2.11. Prove the lemma.

It follows that f is a conformal transformation iff for any point p and tangent vectors v and w, and for any curves γ_v and γ_w with $\dot\gamma_v(0) = v$, $\dot\gamma_w(0) = w$, the curves $f \circ \gamma_v$ and $f \circ \gamma_w$ intersect in the same angle as γ_v and γ_w. This corresponds to the standard interpretation of conformality.

The two dimensional case is special because of the classical fact that orientation preserving conformal maps are holomorphic. The proof is given for completeness.

LEMMA 2.6. *(Conformal = holomorphic) Let Ω and $\tilde\Omega$ be open sets in $\mathbb{R}^2$. An orientation preserving map $f : (\Omega, e) \to (\tilde\Omega, e)$ is conformal iff it is holomorphic and bijective.*

PROOF. We use complex notation and write $z = x+iy$, $f = u+iv$. If f is conformal then it is bijective and $f^*e = ce$. The last condition means that for all $z \in \Omega$ and for $v, w \in \mathbb{R}^2$,

$$c(z)v \cdot w = (f_*v)\cdot(f_*w) = Df(z)v \cdot Df(z)w = Df(z)^t Df(z) v \cdot w.$$

Since $Df(z) = \left(\begin{smallmatrix} u_x & u_y \\ v_x & v_y \end{smallmatrix}\right)$, this implies

$$\begin{pmatrix} u_x^2 + v_x^2 & u_x u_y + v_x v_y \\ u_x u_y + v_x v_y & u_y^2 + v_y^2 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$

Thus the vectors $(u_x \ v_x)^t$ and $(u_y \ v_y)^t$ are orthogonal and have the same length. Since f is orientation preserving so $\det Df > 0$, we must have

$$u_x = v_y, \quad u_y = -v_x.$$

This shows that f is holomorphic. The converse follows by running the argument backwards. $\square$

It follows from the existence of *isothermal coordinates* that any 2D Riemannian manifold is locally conformal to a set in Euclidean space. The conformal structure of manifolds with dimension $n \geq 3$ is much more complicated. However, the model spaces are locally conformally Euclidean.

LEMMA 2.7. (1) *Let (S^n, g) be the unit sphere in $\mathbb{R}^{n+1}$ with its induced metric, and let $N = e_{n+1}$ be the north pole. Then the stereographic projection*

$$f : (S^n \smallsetminus \{N\}, g) \to (\mathbb{R}^n, e), \quad f(y, y_{n+1}) := \frac{y}{1 - y_{n+1}}$$

is a conformal transformation.

(2) *Hyperbolic space (H^n, g) where H^n is the unit ball B in $\mathbb{R}^n$ and $g_{jk}(x) = \frac{4}{(1-|x|^2)^2}\delta_{jk}$, is conformal to (B, e).*

EXERCISE 2.12. Prove the lemma.

Finally, we mention Liouville's theorem which characterizes all conformal transformations in $\mathbb{R}^n$ for $n \geq 3$. This result shows that up to translation, scaling, and rotation, the only conformal transformations are the identity map and Kelvin transform (this is in contrast to the 2D case where there is a rich family of conformal transformations, the holomorphic bijective maps). See [**9**] for a proof.

THEOREM. *(Liouville) If $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^n$ with $n \geq 3$, then an orientation preserving diffeomorphism $f : (\Omega, e) \to (\tilde{\Omega}, e)$ is conformal iff*

$$f(x) = \alpha A h(x - x_0) + b$$

where $\alpha \in \mathbb{R}$, A is an $n \times n$ orthogonal matrix, $h(x) = x$ or $h(x) = \frac{x}{|x|^2}$, $x_0 \in \mathbb{R}^n \smallsetminus \Omega$, and $b \in \mathbb{R}^n$.

2.3. Laplace-Beltrami operator

Definition. In this section we will see that on any Riemannian manifold there is a canonical second order elliptic operator, called the Laplace-Beltrami operator, which is an analog of the usual Laplacian in $\mathbb{R}^n$.

MOTIVATION. Let first Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and consider the Laplace operator

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

Solutions of the equation $\Delta u = 0$ are called harmonic functions, and by standard results for elliptic PDE [**5**, Section 6], for any $f \in H^1(\Omega)$ there is a unique solution $u \in H^1(\Omega)$ of the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \quad u = f & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The last line means that $u - f \in H_0^1(\Omega)$.

One way to produce the solution of (2.3) is based on variational methods and Dirichlet's principle [**5**, Section 2]. We define the Dirichlet energy

$$E(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx, \qquad v \in H^1(\Omega).$$

If we define the admissible class

$$\mathcal{A}_f := \{v \in H^1(\Omega)\,;\, v = f \text{ on } \partial\Omega\},$$

then the solution of (2.3) is the unique function $u \in \mathcal{A}_f$ which minimizes the Dirichlet energy:

$$E(u) \leq E(v) \quad \text{for all } v \in \mathcal{A}_f.$$

The heuristic idea is that the solution of (2.3) represents a physical system in equilibrium, and therefore should minimize a suitable energy functional. The point is that one can start from the energy functional $E(\,\cdot\,)$ and conclude that any minimizer u must satisfy $\Delta u = 0$, which gives another way to define the Laplace operator.

From this point on, let (M, g) be a compact Riemannian manifold with smooth boundary. Although there is no obvious analog of the coordinate definition of Δ in $\mathbb{R}^n$, there is a natural analog of the Dirichlet energy. It is given by

$$E(v) := \frac{1}{2}\int_M |dv|^2 \, dV, \qquad v \in H^1(M).$$

Here $|dv|$ is the Riemannian length of the 1-form dv, and dV is the volume form.

We wish to find a differential equation which is satisfied by minimizers of $E(\,\cdot\,)$. Suppose $u \in H^1(M)$ is a minimizer which satisfies

$E(u) \leq E(u+t\varphi)$ for all $t \in \mathbb{R}$ and all $\varphi \in C_c^\infty(M^{\mathrm{int}})$. We have

$$\begin{aligned} E(u+t\varphi) &= \frac{1}{2}\int_M \langle d(u+t\varphi), d(u+t\varphi)\rangle \, dV \\ &= E(u) + t\int_M \langle du, d\varphi\rangle \, dV + t^2 E(\varphi). \end{aligned}$$

Since $I_\varphi(t) := E(u+t\varphi)$ is a smooth function of t for fixed φ, and since $I_\varphi(0) \leq I_\varphi(t)$ for $|t|$ small, we must have $I_\varphi'(0) = 0$. This shows that if u is a minimizer, then

$$\int_M \langle du, d\varphi\rangle \, dV = 0$$

for any choice of $\varphi \in C_c^\infty(M^{\mathrm{int}})$. By the properties of the codifferential δ, this implies that

$$\int_M (\delta du)\varphi \, dV = 0$$

for all $\varphi \in C_c^\infty(M^{\mathrm{int}})$. Thus any minimizer u has to satisfy the equation

$$\delta du = 0 \quad \text{in } M.$$

We have arrived at the definition of the Laplace-Beltrami operator.

Definition. If (M, g) is a compact Riemannian manifold (with or without boundary), the Laplace-Beltrami operator is defined by

$$\Delta_g u := -\delta du.$$

The next result implies, in particular, that in Euclidean space Δ_g is just the usual Laplacian.

Lemma 2.8. *In local coordinates*

$$\Delta_g u = |g|^{-1/2}\partial_j(|g|^{1/2}g^{jk}\partial_k u)$$

where, as before, $|g| = \det(g_{jk})$ *is the determinant of* g.

Proof. Follows from the coordinate expression for δ. □

Weak solutions. We move on to the question of finding weak solutions to the Dirichlet problem

$$\left\{ \begin{aligned} -\Delta_g u &= F && \text{in } M, \\ u &= 0 && \text{on } \partial M. \end{aligned} \right. \tag{2.4}$$

Here $F \in H^{-1}(M)$ (thus F is a bounded linear functional on $H_0^1(M)$). By definition, a weak solution is a function $u \in H_0^1(M)$ which satisfies

$$\int_M \langle du, d\varphi \rangle \, dV = F(\varphi) \quad \text{for all } \varphi \in H_0^1(M).$$

We will have use of the following compactness result also later.

THEOREM. *(Rellich-Kondrachov compact embedding theorem) Let (M, g) be a compact Riemannian manifold with smooth boundary. Then the natural inclusion $i : H^1(M) \to L^2(M)$ is a compact operator.*

PROOF. See [**5**, Chapter 5] for the Euclidean case and [**20**] for the Riemannian case. □

The solvability of (2.4) will be a consequence of the following inequality.

THEOREM. *(Poincaré inequality) There is $C > 0$ such that*

$$\|u\|_{L^2(M)} \leq C\|du\|_{L^2(M)}, \qquad u \in H_0^1(M).$$

PROOF. Suppose the claim is false. Then there is a sequence $(u_k)_{k=1}^{\infty}$ with $u_k \in H_0^1(M)$ and

$$\|u_k\|_{L^2(M)} > k\|du_k\|_{L^2(M)}.$$

Letting $v_k = u_k/\|u_k\|_{L^2(M)}$, we have $\|v_k\|_{L^2(M)} = 1$ and

$$\|dv_k\|_{L^2(M)} < \frac{1}{k}.$$

Thus (v_k) is a bounded sequence in $H_0^1(M)$, and therefore it has a subsequence (also denoted by (v_k)) which converges weakly to some $v \in H_0^1(M)$. The compact embedding $H^1(M) \hookrightarrow L^2(M)$ implies that

$$v_k \to v \quad \text{in } L^2(M).$$

It follows that $dv_k \to dv$ in $H^{-1}(M)$. But also $dv_k \to 0$ in $L^2(M)$, and uniqueness of limits shows that $dv = 0$. Now any function $v \in H^1(M)$ with $dv = 0$ must be constant on each connected component of M (this follows from the corresponding result in $\mathbb{R}^n$), and since $v \in H_0^1(M)$ we get that $v = 0$. This contradicts the fact that $\|v_k\|_{L^2(M)} = 1$. □

It follows from the Poincaré inequality that for $u \in H_0^1(M)$,

$$\|du\|_{L^2(M)}^2 \leq \|u\|_{H^1(M)}^2 = \|u\|_{L^2(M)}^2 + \|du\|_{L^2(M)}^2 \leq C\|du\|_{L^2(M)}^2.$$

Consequently the norms $\|\cdot\|_{H^1(M)}$ and $\|d\cdot\|_{L^2(M)}$ are equivalent norms on $H^1_0(M)$ (they induce the same topology). We can now prove the solvability of the Dirichlet problem.

Proposition 2.9. *(Existence of weak solutions) The problem* (2.4) *has a unique weak solution $u \in H^1_0(M)$ for any $F \in H^{-1}(M)$. The solution operator*

$$G : H^{-1}(M) \to H^1_0(M), \quad F \mapsto u,$$

is a bounded linear operator.

Proof. Consider the bilinear form

$$B[u,v] := \int_M \langle du, dv\rangle \, dV, \qquad u, v \in H^1_0(M).$$

This satisfies $B[u,v] = B[v,u]$, $|B[u,u]| \leq \|u\|_{H^1_0(M)} \|v\|_{H^1_0(M)}$, and

$$B[u,u] = \int_M |du|^2 \, dV = \|du\|^2_{L^2(M)} \geq C\|u\|^2_{H^1(M)}$$

by using the equivalent norms on $H^1_0(M)$. Thus $H^1_0(M)$ equipped with the inner product $B[\,\cdot\,,\,\cdot\,]$ is the same Hilbert space as $H^1_0(M)$ equipped with the usual inner product $(\,\cdot\,,\,\cdot\,)_{H^1(M)}$. Since F is an element of the dual of $H^1_0(M)$, the Riesz representation theorem shows that there is a unique $u \in H^1_0(M)$ with

$$B[u, \varphi] = F(\varphi), \qquad \varphi \in H^1_0(M).$$

This is the required unique weak solution. Writing $u = GF$, the boundedness of G follows from the estimate $\|u\|_{H^1(M)} \leq \|F\|_{H^{-1}(M)}$ also given by the Riesz representation theorem. $\square$

Corollary 2.10. *(Existence of weak solutions) The problem*

$$\left\{ \begin{array}{rl} -\Delta_g u = 0 & \textit{in } M, \\ u = f & \textit{on } \partial M. \end{array} \right. \tag{2.5}$$

has a unique weak solution $u \in H^1(M)$ for any $f \in H^1(M)$, and the solution satisfies $\|u\|_{H^1(M)} \leq C\|f\|_{H^1(M)}$.

Proof. Let $F = \Delta_g f \in H^{-1}(M)$ (one defines $F(\varphi) := -(df, d\varphi)$). Then (2.5) is equivalent with

$$\left\{ \begin{array}{rl} -\Delta_g (u - f) = F & \text{in } M, \\ u - f = 0 & \text{on } \partial M. \end{array} \right.$$

This has a unique solution $u_0 = GF$ with $\|u_0\|_{H^1(M)} \leq C\|f\|_{H^1(M)}$, and one can take $u = u_0 + f$. $\square$

Spectral theory. Combined with the spectral theorem for compact operators, the previous results show that the spectrum of $-\Delta_g$ consists of a discrete set of eigenvalues and there is an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of $-\Delta_g$.

PROPOSITION 2.11. *(Spectral theory for $-\Delta_g$) Let (M, g) be a compact Riemannian manifold with smooth boundary. There exist numbers $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and an orthonormal basis $\{\phi_l\}_{l=1}^\infty$ of $L^2(M)$ such that*

$$\begin{cases} -\Delta_g \phi_l = \lambda_l \phi_l & \text{in } M, \\ \phi_l \in H_0^1(M). \end{cases}$$

We write $\mathrm{Spec}(-\Delta_g) = \{\lambda_1, \lambda_2, \ldots\}$*. If $\lambda \notin \mathrm{Spec}(-\Delta_g)$, then $-\Delta_g - \lambda$ is an isomorphism from $H_0^1(M)$ onto $H^{-1}(M)$.*

Before giving the proof, we note that by standard Hilbert space theory any function $f \in L^2(M)$ can be written as an L^2-convergent *Fourier series*

$$f = \sum_{l=1}^\infty (f, \phi_l)_{L^2(M)} \phi_l$$

where (f, ϕ_l) is the lth *Fourier coefficient*. These eigenfunction (or Fourier) expansions can sometimes be used as a substitute in M for the Fourier transform in Euclidean space, as we will see in Chapter 4.

PROOF OF PROPOSITION 2.11. Let $G : H^{-1}(M) \to H_0^1(M)$ be the solution operator from Proposition 2.9. By compact embedding, we have that $G : L^2(M) \to L^2(M)$ is compact. It is also self-adjoint and positive semidefinite, since for $f, h \in L^2(M)$ (with $u = Gf$)

$$\begin{aligned} (Gf, h) &= (u, -\Delta_g Gh) = (du, dGh) = (-\Delta_g u, Gh) = (f, Gh), \\ (Gf, f) &= (Gf, -\Delta_g Gf) = (dGf, dGf) \geq 0. \end{aligned}$$

By the spectral theorem for compact operators, there exist $\mu_1 \geq \mu_2 \geq \ldots$ with $\mu_j \to 0$ and $\phi_l \in L^2(M)$ with $G\phi_l = \mu_l \phi_l$ such that $\{\phi_l\}_{l=1}^\infty$ is an orthonormal basis of $L^2(M)$. Note that 0 is not in the spectrum of G, since $Gf = 0$ implies $f = 0$. Taking $\lambda_l = \frac{1}{\mu_l}$ gives $-\Delta_g \phi_l = \lambda_l \phi_l$. If $\lambda \neq \lambda_l$ for all l then for $F \in H^{-1}(M)$,

$$(-\Delta_g - \lambda)u = F \Leftrightarrow u = G(F + \lambda u) \Leftrightarrow (\frac{1}{\lambda}\mathrm{Id} - G)u = \frac{1}{\lambda} GF.$$

Since $\frac{1}{\lambda} \neq \mu_l$ for all l, $\frac{1}{\lambda}\mathrm{Id} - G$ is invertible and we see that $-\Delta_g - \lambda$ is bijective and bounded, therefore an isomorphism. □

We conclude the section with an analog of Proposition 2.11 where the Laplace-Beltrami operator is replaced by the Schrödinger operator $-\Delta_g + V$. The proof is the same except for minor modifications and is left as an exercise. The main point is that for λ outside the discrete set $\mathrm{Spec}(-\Delta_g + V)$, this result implies unique solvability for the Dirichlet problem

$$\left\{ \begin{array}{rl} (-\Delta_g + V - \lambda)u = 0 & \text{in } M, \\ u = f & \text{on } \partial M \end{array} \right.$$

with the norm estimate $\|u\|_{H^1(M)} \leq C\|f\|_{H^1(M)}$.

Proposition 2.12. *(Spectral theory for $-\Delta_g + V$) Let (M, g) be a compact Riemannian manifold with smooth boundary, and assume that $V \in L^\infty(M)$ is real valued. There exist numbers $\lambda_1 \leq \lambda_2 \leq \ldots$ and an orthonormal basis $\{\psi_l\}_{l=1}^\infty$ of $L^2(M)$ such that*

$$\left\{ \begin{array}{rl} (-\Delta_g + V)\psi_l = \lambda_l \psi_l & \textit{in } M, \\ \psi_l \in H_0^1(M). & \end{array} \right.$$

We write $\mathrm{Spec}(-\Delta_g + V) = \{\lambda_1, \lambda_2, \ldots\}$. *If* $\lambda \notin \mathrm{Spec}(-\Delta_g + V)$, *then* $-\Delta_g + V - \lambda$ *is an isomorphism from* $H_0^1(M)$ *onto* $H^{-1}(M)$.

Exercise 2.13. Prove this result by first showing an analog of Proposition 2.9 where $-\Delta_g$ is replaced by $-\Delta_g + V + k_0$ for k_0 sufficiently large, and then by following the proof of Proposition 2.11 where G is replaced by the inverse operator for $-\Delta_g + V + k_0$.

2.4. DN map

Definition. We now rigorously define the Dirichlet-to-Neumann map, or DN map for short, discussed in the introduction. Let (M, g) be a compact manifold with smooth boundary, and let $V \in L^\infty(M)$. Proposition 2.12 shows that the Dirichlet problem

$$\left\{ \begin{array}{rl} (-\Delta_g + V)u = 0 & \text{in } M, \\ u = f & \text{on } \partial M \end{array} \right. \tag{2.6}$$

has a unique solution $u \in H^1(M)$ for any $f \in H^1(M)$, provided that 0 is not a Dirichlet eigenvalue (meaning that $0 \notin \mathrm{Spec}(-\Delta_g + V)$). We make the standing assumption that all Schrödinger operators are such that

$$0 \text{ is not a Dirichlet eigenvalue of } -\Delta_g + V.$$

As mentioned in the introduction, it would be easy to remove this assumption by using so called Cauchy data sets instead of the DN map.

If 0 is not a Dirichlet eigenvalue, then (2.6) is uniquely solvable for any $f \in H^1(M)$. If $f \in H_0^1(M)$ then $u = 0$ is a solution (since then $u - f \in H_0^1(M)$), which means that the solution with boundary value f coincides with the solution with boundary value $f + \varphi$ where $\varphi \in H_0^1(M)$. Motivated by this, we define the quotient space

$$H^{1/2}(\partial M) := H^1(M)/H_0^1(M).$$

This is a Hilbert space which can be identified with a space of functions on ∂M which have 1/2 derivatives in $L^2(\partial M)$, but the abstract setup will be enough for us. We also define $H^{-1/2}(\partial M)$ as the dual space of $H^{1/2}(\partial M)$.

By the above discussion, the Dirichlet problem (2.6) is well posed for boundary values $f \in H^{1/2}(M)$. Denoting the solution by u_f, the DN map is formally defined as the map

$$\Lambda_{g,V} : f \mapsto \partial_\nu u_f|_{\partial M}.$$

Here, for sufficiently smooth u, the normal derivative is defined by

$$\partial_\nu u|_{\partial M} := \langle \nabla u, \nu \rangle|_{\partial M}.$$

To find a rigorous definition of Λ_g we will use an integration by parts formula.

Theorem. *(Green's formula) If $u, v \in C^2(M)$ then*

$$\int_{\partial M} (\partial_\nu u) v \, dS = \int_M (\Delta_g u) v \, dV + \int_M \langle du, dv \rangle \, dV.$$

Exercise 2.14. Prove this formula by using Stokes' theorem.

Let now $f, h \in H^{1/2}(\partial M)$, let u_f be the solution of (2.6), and let e_h be any function in $H^1(M)$ with $e_h|_{\partial M} = h$ (with natural interpretations). Then, again purely formally,

$$\begin{aligned}\langle \Lambda_{g,V} f, h\rangle &= \int_{\partial M} (\partial_\nu u_f) e_h \, dS = \int_M (\Delta_g u_f) e_h \, dV + \int_M \langle du_f, de_h\rangle \, dV \\ &= \int_M \left[\langle du_f, de_h\rangle + V u_f e_h\right] dV.\end{aligned}$$

We have finally arrived at the precise definition of $\Lambda_{g,V}$.

Definition. $\Lambda_{g,V}$ is the linear map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ defined via the bilinear form

$$\langle \Lambda_{g,V} f, h\rangle = \int_M \left[\langle du_f, de_h\rangle + V u_f e_h\right] dV, \quad f, h \in H^{1/2}(\partial M),$$

where u_f and e_h are as above.

Exercise 2.15. Prove that the bilinear form indeed gives a well defined map $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

The DN map is also self-adjoint:

Lemma 2.13. *If V is real valued, then*

$$\langle \Lambda_{g,V} f, h\rangle = \langle f, \Lambda_{g,V} h\rangle, \quad f, h \in H^{1/2}(\partial M).$$

Exercise 2.16. Prove the lemma.

Integral identity. The main point in this section is an integral identity which relates the difference of two DN maps to an integral over M involving the difference of two potentials. This identity is the starting point for recovering interior information (the potentials in M) from boundary measurements (the DN maps on ∂M).

Proposition 2.14. *(Integral identity) Let (M, g) be a compact Riemannian manifold with smooth boundary, and let $V_1, V_2 \in L^\infty(M)$ be real valued. Then*

$$\langle (\Lambda_{g,V_1} - \Lambda_{g,V_2}) f_1, f_2\rangle = \int_M (V_1 - V_2) u_1 u_2 \, dV, \quad f_1, f_2 \in H^{1/2}(\partial M),$$

where $u_j \in H^1(M)$ are the solutions of $(-\Delta_g + V_j) u_j = 0$ in M with $u_j|_{\partial M} = f_j$.

PROOF. By definition and by self-adjointness of Λ_{g,V_2},

$$\langle \Lambda_{g,V_1} f_1, f_2 \rangle = \int_M [\langle du_1, du_2 \rangle + V_1 u_1 u_2] \, dV,$$

$$\langle \Lambda_{g,V_2} f_1, f_2 \rangle = \langle f_1, \Lambda_{g,V_2} f_2 \rangle = \int_M [\langle du_1, du_2 \rangle + V_2 u_1 u_2] \, dV.$$

The result follows by substracting the two identities. □

In this text we are interested in uniqueness results, where one would like to show that $\Lambda_{g,V_1} = \Lambda_{g,V_2}$ implies $V_1 = V_2$. For this purpose, the following corollary is appropriate. It shows that if two DN maps coincide, then the integral of the difference of potentials against the product of *any* two solutions (with no requirements for their boundary values) vanishes.

COROLLARY 2.15. *(Integral identity) Let (M, g) be a compact Riemannian manifold with smooth boundary, and let $V_1, V_2 \in L^\infty(M)$ be real valued. If $\Lambda_{g,V_1} = \Lambda_{g,V_2}$, then*

$$\int_M (V_1 - V_2) u_1 u_2 \, dV = 0$$

for any $u_j \in H^1(M)$ which satisfy $(-\Delta_g + V_j)u_j = 0$ in M.

2.5. Geodesics and covariant derivative

In this section we let (M, g) be a connected Riemannian manifold *without* boundary (for our purposes, geodesics and the Levi-Civita connection on manifolds with boundary can be defined by embedding into a compact manifold without boundary).

Lengths of curves. For the analysis of the Calderón problem on manifolds we will need to introduce some basic properties of geodesics. These are locally length minimizing curves on (M, g), so we begin by discussing lengths of curves.

DEFINITION. A smooth map $\gamma : [a, b] \to M$ whose tangent vector $\dot{\gamma}(t)$ is always nonzero is called a *regular curve*. The *length* of γ is defined by

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)| \, dt.$$

The length of a piecewise regular curve is defined as the sum of lengths of the regular parts. The *Riemannian distance* between two points $p, q \in M$ is defined by

$$d(p,q) := \inf\{L(\gamma)\,;\, \gamma : [a,b] \to M \text{ is a piecewise regular curve with } \gamma(a) = p \text{ and } \gamma(b) = q\}.$$

EXERCISE 2.17. Show that $L(\gamma)$ is independent of the way the curve γ is parametrized, and that we may always parametrize γ by *arc length* so that $|\dot{\gamma}(t)| = 1$ for all t.

EXERCISE 2.18. Show that d is a metric distance function on M, and that (M,d) is a metric space whose topology is the same as the original topology on M.

Geodesic equation. We now wish to show that any length minimizing curve satisfies a certain ordinary differential equation. Suppose that $\gamma : [a,b] \to M$ is a length minimizing curve between two points p and q parametrized by arc length, and let $\gamma_s : [a,b] \to M$ be a family of curves from p to q such that $\gamma_0(t) = \gamma(t)$ and $\Gamma(s,t) := \gamma_s(t)$ depends smoothly on $s \in (-\varepsilon, \varepsilon)$ and on $t \in [a,b]$. We assume for simplicity that each γ_s is regular and contained in a coordinate neighborhood of M, and write $x_s(t) = (x_s^1(t), \ldots, x_s^n(t))$ and $x(t) = x_0(t)$ instead of $\gamma_s(t)$ and $\gamma(t)$ in local coordinates.

LEMMA 2.16. *The length minimizing curve $x(t)$ satisfies the so called* geodesic equation

$$\ddot{x}^l(t) + \Gamma_{jk}^l(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \quad 1 \leq l \leq n,$$

where Γ_{jk}^l is the Christoffel symbol

$$\Gamma_{jk}^l = \frac{1}{2} g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}).$$

PROOF. Since γ minimizes length from p to q, we have

$$L(\gamma_0) \leq L(\gamma_s), \qquad s \in (-\varepsilon, \varepsilon).$$

Define

$$I(s) := L(\gamma_s) = \int_a^b (g_{pq}(x_s(t))\dot{x}_s^p(t)\dot{x}_s^q(t))^{1/2}\,dt.$$

Since I is smooth and $I(0) \leq I(s)$ for $|s| < \varepsilon$, we must have $I'(0) = 0$. To prepare for computing the derivative, define two vector fields

$$T(t) := \partial_t x_s(t)|_{s=0}, \quad V(t) := \partial_s x_s(t)|_{s=0}.$$

Using that $|\dot{\gamma}_0(t)| = 1$ and (g_{jk}) is symmetric, we have

$$I'(0) = \frac{1}{2}\int_a^b (\partial_r g_{pq}(x(t))V^r(t)T^p(t)T^q(t) + 2g_{pq}(x(t))\dot{V}^p(t)T^q(t))\,dt.$$

Integrating by parts in the last term, this shows that

$$I'(0) = \int_a^b \left[\frac{1}{2}\partial_r g_{pq}(x)T^pT^q - \partial_m g_{rq}(x)T^mT^q - g_{rq}(x)\dot{T}^q\right] V^r\,dt.$$

The last expression vanishes for all possible vector fields $V(t)$ obtained as $\partial_s x_s(t)|_{s=0}$. It can be seen that any vector field with $V(a) = V(b) = 0$ arises as $V(t)$ for some family of curves $\gamma_s(t)$. This implies that

$$\frac{1}{2}\partial_r g_{pq}(x)T^pT^q - \partial_m g_{rq}(x)T^mT^q - g_{rq}(x)\dot{T}^q = 0,\ t \in [a,b], 1 \leq r \leq n.$$

Multiplying this by g^{lr} and summing over r, and using that

$$\partial_m g_{rq}(x)T^mT^q = \frac{1}{2}(\partial_m g_{rq}(x) + \partial_q g_{rm}(x))T^mT^q,$$

gives the geodesic equation upon relabeling indices. □

Covariant derivative. It would be possible to develop the theory of geodesics based on the ODE derived in Lemma 2.16. However, it will be very useful to be able to do computations such as those in Lemma 2.16 in an invariant way, without resorting to local coordinates. For this purpose we want to be able to take derivatives of vector fields in a way which is compatible with the Riemannian inner product $\langle\,\cdot\,,\,\cdot\,\rangle$.

We first recall the commutator of vector fields. Any vector field $X \in C^\infty(M,TM)$ gives rise to a first order differential operator $X : C^\infty(M) \to C^\infty(M)$ by

$$Xf(p) = X(p)f.$$

If X and Y are vector fields, their commutator $[X,Y]$ is the differential operator acting on smooth functions by

$$[X,Y]f := X(Yf) - Y(Xf).$$

The commutator of two vector fields is itself a vector field.

The next result is sometimes called the fundamental lemma of Riemannian geometry.

Theorem. *(Levi-Civita connection) On any Riemannian manifold (M,g) there is a unique $\mathbb{R}$-bilinear map*

$$D : C^\infty(M,TM) \times C^\infty(M,TM) \to C^\infty(M,TM), \quad (X,Y) \mapsto D_XY,$$

which satisfies

(1)	$D_{fX}Y = fD_XY$	*(linearity)*
(2)	$D_X(fY) = fD_XY + (Xf)Y$	*(Leibniz rule)*
(3)	$D_XY - D_YX = [X,Y]$	*(symmetry)*
(4)	$X\langle Y,Z\rangle = \langle D_XY, Z\rangle + \langle Y, D_XZ\rangle$	*(metric connection).*

Here X, Y, Z are vector fields and f is a smooth function on M.

Proof. See [**12**]. □

The map D is called the *Levi-Civita connection* of (M,g). The expression D_XY is called the *covariant derivative* of the vector field Y in direction X.

Example. In $(\mathbb{R}^n, e)$ the Levi-Civita connection is given by

$$D_XY = X^j(\partial_jY^k)\partial_k.$$

This is just the natural derivative of Y in direction X.

Example. On a general manifold (M,g), one has

$$D_XY = X^j(\partial_jY^k)\partial_k + X^jY^k\Gamma_{jk}^l\partial_l$$

where Γ_{jk}^l are the Christoffel symbols from Lemma 2.16, and they also satisfy

$$D_{\partial_j}\partial_k = \Gamma_{jk}^l\partial_l.$$

Covariant derivative of tensors. At this point we will define the connection and covariant derivatives also for other tensor fields. Let X be a vector field on M. The covariant derivative of 0-tensor fields is given by

$$D_Xf := Xf.$$

For k-tensor fields u, the covariant derivative is defined by

$$D_Xu(Y_1,\ldots,Y_k) := X(u(Y_1,\ldots,Y_k)) - \sum_{j=1}^k u(Y_1,\ldots,D_XY_j,\ldots,Y_k).$$

Exercise 2.19. Show that these formulas give a well defined covariant derivative

$$D_X : C^\infty(M,T^kM) \to C^\infty(M,T^kM).$$

Example. The main example of the above construction is the covariant derivative of 1-forms, which is uniquely specified by the identity

$$D_{\partial_j} dx^k = -\Gamma_{jl}^k dx^l.$$

By using D_X on tensors, it is possible to define the *total covariant derivative* as the map

$$D : C^\infty(M, T^k M) \to C^\infty(M, T^{k+1} M),$$
$$Du(X, Y_1, \ldots, Y_k) := D_X u(Y_1, \ldots, Y_k).$$

Example. On 0-forms $Df = df$.

Example. The most important use for the total covariant derivative in these notes is the *covariant Hessian.* If f is a smooth function, then the covariant Hessian of f is

$$\mathrm{Hess}(f) := D^2 f.$$

In local coordinates it is given by

$$D^2 f = (\partial_j \partial_k f - \Gamma_{jk}^l \partial_l f)\, dx^j \otimes dx^k.$$

Finally, we mention that the total covariant derivative can be used to define higher order Sobolev spaces invariantly on a Riemannian manifold.

Definition. If $k \geq 0$, consider the inner product on $C^\infty(M)$ given by

$$(u, v)_{H^k(M)} := \sum_{j=0}^k (D^j u, D^j v)_{L^2(M)}.$$

Here the L^2 norm is the natural one using the inner product on tensors. The Sobolev space $H^k(M)$ is defined to be the completion of $C^\infty(M)$ with respect to this inner product.

Geodesics. Let us return to length minimizing curves. If $\gamma : [a, b] \to M$ is a curve and $X : [a, b] \to TM$ is a smooth vector field along γ (meaning that $X(t) \in T_{\gamma(t)} M$), we define the derivative of X along γ by

$$D_{\dot\gamma} X := D_{\dot\gamma} \tilde X$$

where $\tilde X$ is any vector field defined in a neighborhood of $\gamma([a, b])$ such that $\tilde X_{\gamma(t)} = X_{\gamma(t)}$. It is easy to see that this does not depend on the choice of $\tilde{X}$. The relation to geodesics now comes from the fact that in local coordinates, if $\gamma(t)$ corresponds to $x(t)$,

$$\begin{aligned} D_{\dot\gamma}\dot\gamma &= D_{\dot x^j \partial_j}(\dot x^k \partial_k) \\ &= (\ddot x^l + \Gamma^l_{jk}(x)\dot x^j \dot x^k)\partial_l. \end{aligned}$$

Thus the geodesic equation is satisfied iff $D_{\dot\gamma}\dot\gamma = 0$. We now give the precise definition of a geodesic.

DEFINITION. A regular curve γ is called a geodesic if $D_{\dot\gamma}\dot\gamma = 0$.

The arguments above give evidence to the following result, which is proved for instance in [**12**].

THEOREM. *(Geodesics minimize length) If γ is a piecewise regular length minimizing curve from p to q, then γ is regular and $D_{\dot\gamma}\dot\gamma = 0$. Conversely, if γ is a regular curve and $D_{\dot\gamma}\dot\gamma = 0$, then γ minimizes length at least locally.*

We next list some basic properties of geodesics.

THEOREM. *(Properties of geodesics) Let (M, g) be a Riemannian manifold without boundary. Then*

1. *for any $p \in M$ and $v \in T_pM$, there is an open interval I containing 0 and a geodesic $\gamma_v : I \to M$ with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$,*
2. *any two geodesics with $\gamma_1(0) = \gamma_2(0)$ and $\dot\gamma_1(0) = \dot\gamma_2(0)$ agree in their common domain,*
3. *any geodesic satisfies $|\dot\gamma(t)| = const$,*
4. *if M is compact then any geodesic γ can be uniquely extended as a geodesic defined on all of $\mathbb{R}$.*

EXERCISE 2.20. Prove this theorem by using the existence and uniqueness of solutions to ordinary differential equations.

By (3) in the theorem, we may (and will) always assume that geodesics are parametrized by arc length and satisfy $|\dot\gamma| = 1$. Part (4) says that the maximal domain of any geodesic on a closed manifold is $\mathbb{R}$, where the maximal domain is the largest interval to which the geodesic can be extended. We will always assume that the geodesics are defined on their maximal domain.

Normal coordinates. The following important concept enables us to parametrize a manifold locally by its tangent space.

Definition. If $p \in M$ let $\mathcal{E}_p := \{v \in T_pM\,;\, \gamma_v \text{ is defined on } [0,1]\}$, and define the *exponential map*

$$\exp_p : \mathcal{E}_p \to M, \quad \exp_p(v) = \gamma_v(1).$$

This is a smooth map and satisfies $\exp_p(tv) = \gamma_v(t)$. Thus, the exponential map is obtained by following radial geodesics starting from the point p. This parametrization also gives rise to a very important system of coordinates on Riemannian manifolds.

Theorem. *(Normal coordinates) For any $p \in M$, $\exp_p$ is a diffeomorphism from some neighborhood V of 0 in T_pM onto a neighborhood of p in M. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM and we identify T_pM with $\mathbb{R}^n$ via $v^j e_j \leftrightarrow (v^1, \ldots, v^n)$, then there is a coordinate chart (U, φ) such that $\varphi = \exp_p^{-1} : U \to \mathbb{R}^n$ and*

1. *$\varphi(p) = 0$,*
2. *if $v \in T_pM$ then $\varphi(\gamma_v(t)) = (tv^1, \ldots, tv^n)$,*
3. *one has*

$$g_{jk}(0) = \delta_{jk}, \quad \partial_l g_{jk}(0) = 0, \quad \Gamma^l_{jk}(0) = 0.$$

Proof. See [**12**]. □

The local coordinates in the theorem are called *normal coordinates* at p. In these coordinates geodesics through p correspond to rays through the origin. Further, by (3) the metric and its first derivatives have a simple form at 0. This fact is often exploited when proving an identity where both sides are invariantly defined, and thus it is enough to verify the identity in some suitable coordinate system. The properties given in (3) sometimes simplify these local coordinate computations dramatically.

Finally, we will need the fact that when switching to polar coordinates in a normal coordinate system, the metric has special form in a full neighborhood of 0 instead of just at the origin.

Theorem. *(Polar normal coordinates) Let (U, φ) be normal coordinates at p. If (r, θ) are the corresponding polar coordinates (thus $r(q) = |\varphi(q)| > 0$ and $\theta(q)$ is the corresponding direction in S^{n-1}), then* *the metric has the form*

$$(g_{jk}(r,\theta)) = \begin{pmatrix} 1 & 0 \\ 0 & g_{\alpha\beta}(r,\theta) \end{pmatrix}.$$

This means that $|\partial/\partial r| = 1$, $\langle \partial/\partial r, \partial/\partial\theta\rangle = 0$, *and* $r(q) = d(p,q)$.

CHAPTER 3

Limiting Carleman weights

In this chapter we will establish a starting point for solving some of the problems mentioned in the introduction. The approach taken here is to construct special solutions to the Schrödinger equation (or special harmonic functions if there is no potential) in (M, g), in such a way that the products of these special solutions are dense in $L^1(M)$.

The exact form of the special solutions is motivated by developments in $\mathbb{R}^n$, where harmonic exponential functions $e^{\rho \cdot x}$ with $\rho \in \mathbb{C}^n$ and $\rho \cdot \rho = 0$ have been successful in the solution of inverse problems. On a Riemannian manifold there is no immediate analog for the linear phase function $\rho \cdot x$ (one can always find such a function in local coordinates, but not globally in general). We will instead look for general phase functions φ which are expected to have desirable properties for the purposes of constructing special solutions. Such phase functions will be called *limiting Carleman weights* (LCWs).

The main result is a geometric characterization of those manifolds which admit LCWs. It makes use of the crucial fact that the existence of LCWs only depends on the conformal class of the manifold. The result is stated in terms of the existence of a parallel vector field in some conformal manifold.

THEOREM 3.1. *(Manifolds which admit LCWs) Let (M, g) be a simply connected open Riemannian manifold. Then (M, g) admits an LCW iff some conformal multiple of g admits a parallel unit vector field.*

Intuitively, the geometric condition means that up to a conformal factor there has to be a Euclidean direction on the manifold.

At this point we also mention a few open questions related to the theorem. The notation will be explained below. The first question asks to show that in dimensions $n \geq 3$ most metrics do not admit LCWs even locally (in fact, it would be interesting to prove the existence of even one metric which does not admit LCWs).

QUESTION 3.1. *(Counterexamples) If M is a smooth manifold of dimension $n \geq 3$ and if $p \in M$, show that a generic metric near p does not admit an LCW.* [1]

We will show later that if φ is an LCW, then one has a suitable Carleman estimate for the conjugated Laplace-Beltrami operators $P_{\pm\varphi}$. The next question is asking for a converse.

QUESTION 3.2. *(Carleman estimates imply LCW) If (M, g) is an open manifold and φ is such that for any $M_1 \subset\subset M$ there are $C_0, h_0 > 0$ for which*

$$h\|u\|_{L^2(M_1)} \leq C\|P_{\pm\varphi}u\|_{L^2(M_1)}, \quad u \in C_c^\infty(M_1^{int}),\ 0 < h < h_0,$$

then φ is an LCW. [2]

The last question asks to find an analog in dimensions $n \geq 3$ of the Carleman weights with critical points which have recently been very successful in 2D inverse problems.

QUESTION 3.3. *Find an analog in dimensions $n \geq 3$ of Bukhgeim-type weights φ in 2D manifolds which satisfy a Carleman estimate of the type $h^{3/2}\|u\| \leq C\|P_{\pm\varphi}u\|$ for $u \in C_c^\infty(M^{int})$ and $0 < h < h_0$.*

In this chapter we will mostly follow [**4**, Section 2].

3.1. Motivation and definition

Let (M, g) be a compact Riemannian manifold with boundary, and let $V_1, V_2 \in C^\infty(M)$. As always, we assume that the Dirichlet problems for $-\Delta_g + V_j$ in M are uniquely solvable, so that the DN maps Λ_{g,V_j} are well defined. Assume that $\Lambda_{g,V_1} = \Lambda_{g,V_2}$, that is, the two potentials V_1 and V_2 result in identical boundary measurements. Then we know that

$$\int_M (V_1 - V_2)u_1 u_2 \, dV = 0$$

for any solutions $u_j \in H^1(M)$ which satisfy $(-\Delta_g + V_j)u_j = 0$ in M. To solve the inverse problem of proving that $V_1 = V_2$, it is therefore enough to show that the set of products of solutions

$$\{u_1 u_2\,;\ u_j \in H^1(M) \text{ and } (-\Delta_g + V_j)u_j = 0 \text{ in } M\}$$

[1]A positive answer to this question was recently given in [**14**].

[2]A positive answer was outlined in lectures of Dos Santos Ferreira [**3**].

is dense in $L^1(M)$.

In Euclidean space in dimensions $n \geq 3$, the density of solutions can be proved based on harmonic complex exponentials. The following argument is from [**19**] and is explained in detail in [**17**, Chapter 3].

MOTIVATION. Let $(M, g) = (\overline{\Omega}, e)$ where Ω is a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary. In this setting we have special harmonic functions

$$u_0(x) = e^{\rho \cdot x}, \quad \rho \in \mathbb{C}^n, \ \rho \cdot \rho = 0. \tag{3.1}$$

Clearly $\Delta u_0 = (\rho \cdot \rho) u_0 = 0$. By [**19**], if $|\rho|$ is large there exist solutions to Schrödinger equations which look like these harmonic exponentials and have the form

$$\begin{aligned} u_1 &= e^{\rho \cdot x}(a_1 + r_1), \\ u_2 &= e^{-\rho \cdot x}(a_2 + r_2), \end{aligned}$$

where a_j are certain explicit functions and r_j are correction terms which are small when $|\rho|$ is large, in the sense that $\|r_j\|_{L^2(\Omega)} \leq C/|\rho|$. We have chosen one solution with $e^{\rho \cdot x}$ and the other solution with $e^{-\rho \cdot x}$ so that the exponential factors will cancel in the product $u_1 u_2$, thus making it possible to take the limit as $|\rho| \to \infty$ which will get rid of the correction terms r_j.

The density of products of solutions in this case can be proved as follows. We fix $\xi \in \mathbb{R}^n$ and choose $a_1 = e^{ix \cdot \xi}$, $a_2 = 1$. If $n \geq 3$ then there exists a family of complex vectors ρ with $\rho \cdot \rho = 0$ and $|\rho| \to \infty$ such that solutions with the above properties can be constructed. To show density of the set $\{u_1 u_2\}$ for solutions of this type, we take $V \in L^\infty(\Omega)$ and assume that

$$\int_\Omega V u_1 u_2 \, dx = 0$$

for all u_1 and u_2 as above. Then

$$\int_\Omega V(e^{ix \cdot \xi} + r_1 + e^{ix \cdot \xi} r_2 + r_1 r_2) \, dx = 0.$$

By the L^2 estimates for r_j we may take the limit as $|\rho| \to \infty$, which will imply that $\int_\Omega V e^{ix \cdot \xi} \, dx = 0$. Since this is true for any fixed $\xi \in \mathbb{R}^n$, it follows from the uniqueness of the Fourier transform that $V = 0$ as required.

After having discussed the proof in the Euclidean case, we move on to the setting on Riemannian manifolds and try to see if a similar argument could be achieved. If (M, g) is a compact Riemannian manifold with boundary, we first seek approximate solutions satisfying $\Delta_g u_0 \approx 0$ (in some sense) having the form

$$u_0 = e^{-\varphi/h} m.$$

Here φ is assumed to be a smooth real valued function on M, $h > 0$ will be a small parameter, and $m \in C^\infty(M)$ is some complex function. In the Euclidean case this corresponds to (3.1) by taking $h = 1/|\rho|$, $\varphi(x) = -\mathrm{Re}(\rho/|\rho|) \cdot x$, and $m(x) = e^{\mathrm{Im}(\rho)\cdot x}$.

Loosely speaking, φ will be a limiting Carleman weight if such approximate solutions with weight $\pm\varphi$ can always be converted into exact solutions of $\Delta_g u = 0$ (we can forget the potential V at this point). More precisely, we would like that

$$\begin{cases} \text{for any function } u_0 = e^{\mp\varphi/h} m \in C^\infty(M) \text{ there is a} \\ \text{solution } u = e^{\mp\varphi/h}(m + r) \text{ of } \Delta_g u = 0 \text{ in } M \text{ such that} \\ \|r\|_{L^2(M)} \leq Ch \|e^{\pm\varphi/h} \Delta_g u_0\|_{L^2(M)} \text{ for } h \text{ small.} \end{cases} \tag{3.2}$$

To find conditions on φ which would guarantee that this is possible, we introduce the conjugated Laplace-Beltrami operator

$$P_\varphi := e^{\varphi/h}(-h^2 \Delta_g) e^{-\varphi/h}.$$

Note that if $u = e^{\mp\varphi/h}(m + r)$, then

$$\begin{aligned} \Delta_g u = 0 \ &\Leftrightarrow \ e^{\pm\varphi/h}(-h^2 \Delta_g) e^{\mp\varphi/h}(m + r) = 0 \\ &\Leftrightarrow \ P_{\pm\varphi} r = -P_{\pm\varphi} m. \end{aligned}$$

Thus (3.2) would follow if for any $f \in L^2(M)$ there is a function v satisfying for h small

$$\begin{aligned} P_{\pm\varphi} v &= f \quad \text{in } M, \\ h\|v\|_{L^2(M)} &\leq C\|f\|_{L^2(M)}. \end{aligned}$$

One approach for proving existence of solutions to the last equation, or more generally an inhomogeneous equation $Tv = f$, is to prove uniqueness of solutions to the homogeneous adjoint equation $T^* v = 0$. This follows the general principle

$$\begin{cases} T^* \text{ injective} \\ \text{range of } T^* \text{ closed} \end{cases} \implies T \text{ surjective.}$$

Exercise 3.1. Find out why this principle holds for $m \times n$ matrices, for operators $T = \mathrm{Id} + K$ where K is a compact operator on a Hilbert space, or for bounded operators T between two Hilbert spaces.

Since $P_{\pm\varphi}^* = P_{\mp\varphi}$, injectivity and closed range for the adjoint operator would be a consequence of the *a priori* estimate

$$h\|u\|_{L^2(M)} \leq C\|P_{\pm\varphi}u\|_{L^2(M)}, \quad u \in C_c^\infty(M^{\mathrm{int}}), \ h \text{ small.} \tag{3.3}$$

This is called a *Carleman estimate* (that is, a norm estimate with exponential weights depending on a parameter). Estimates of this type have turned out to be very useful in unique continuation for solutions of partial differential equations, control theory, and inverse problems.

We will look for conditions on φ which would imply the Carleman estimate (3.3). The following decomposition of P_φ into its self-adjoint part A and skew-adjoint part iB will be useful.

Lemma 3.2. *$P_\varphi = A + iB$ where A and B are the formally self-adjoint operators (in the $L^2(M)$ inner product)*

$$\begin{aligned} A &:= -h^2\Delta_g - |d\varphi|^2, \\ B &:= \frac{h}{i}\left(2\langle d\varphi, d\,\cdot\rangle + \Delta_g\varphi\right). \end{aligned}$$

Proof. The quickest way to see this is a computation in local coordinates. We write $D_j = -i\partial_{x_j}$, and note that

$$e^{\varphi/h}hD_je^{-\varphi/h} = hD_j + i\varphi_{x_j}.$$

Then

$$\begin{aligned} P_\varphi u &= e^{\varphi/h}(-h^2\Delta_g)e^{-\varphi/h}u \\ &= |g|^{-1/2}e^{\varphi/h}hD_j(e^{-\varphi/h}|g|^{1/2}g^{jk}e^{\varphi/h}hD_k(e^{-\varphi/h}u)) \\ &= |g|^{-1/2}(hD_j + i\varphi_{x_j})|g|^{1/2}g^{jk}(hD_k + i\varphi_{x_k})u) \\ &= -h^2\Delta_g u + hg^{jk}\varphi_{x_j}u_{x_k} + h|g|^{-1/2}\partial_j(|g|^{1/2}g^{jk}\varphi_{x_k}u) - g^{jk}\varphi_{x_j}\varphi_{x_k}u \\ &= -h^2\Delta_g u + h\left[2\langle d\varphi, du\rangle + (\Delta_g\varphi)u\right] - |d\varphi|^2u. \end{aligned}$$

The result follows immediately upon checking that A and B are formally self-adjoint. $\square$

Exercise 3.2. Check that A and B are formally self-adjoint.

Next we give a basic computation in the proof of a Carleman estimate such as (3.3), evaluating the square of the right hand side.

LEMMA 3.3. *If* $u \in C_c^\infty(M^{int})$ *then*

$$\|P_\varphi u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u,u).$$

PROOF. Since $P_\varphi = A + iB$,

$$\begin{aligned}\|P_\varphi u\|^2 &= (P_\varphi u, P_\varphi u) = ((A+iB)u,(A+iB)u) \\ &= (Au,Au) + i(Bu,Au) - i(Au,Bu) + (Bu,Bu) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A,B]u,u).\end{aligned}$$

We used that A and B are formally self-adjoint. □

Thus $\|P_\varphi u\|^2$ can be written as the sum of two nonnegative terms $\|Au\|^2$ and $\|Bu\|^2$ and a third term which involves the commutator $[A,B] := AB - BA$. The only negative contribution may come from the commutator term. Therefore, a positivity condition for $i[A,B]$ would be helpful for proving the Carleman estimate (3.3) for P_φ. We will state such a positivity condition on the level of principal symbols.

LEMMA 3.4. *The principal symbols of* A *and* B *are*

$$\begin{aligned} a(x,\xi) &:= |\xi|^2 - |d\varphi|^2, \\ b(x,\xi) &:= 2\langle d\varphi, \xi\rangle. \end{aligned}$$

The principal symbol of $i[A,B]$ *is the Poisson bracket* $h\{a,b\}$.

PROOF. The principal symbol of A is obtained by writing A in some local coordinates and by looking at the symbol of the corresponding operator in $\mathbb{R}^n$. But in local coordinates

$$A = g^{jk} hD_j hD_k - g^{jk}\varphi_{x_j}\varphi_{x_k} + h\left[|g|^{-1/2} D_j(|g|^{1/2} g^{jk}) D_k\right].$$

The last term is lower order, hence does not affect the principal symbol. The symbol of $g^{jk} hD_j hD_k - g^{jk}\varphi_{x_j}\varphi_{x_k}$ is $g^{jk}\xi_j\xi_k - g^{jk}\varphi_{x_j}\varphi_{x_k}$, so we may take the invariantly defined function $a(x,\xi) := |\xi|^2 - |d\varphi|^2$ on T^*M as the principal symbol. A similar argument works for B, and the claim for $i[A,B]$ is a general fact. □

Given this information, the positivity condition that we will require of $i[A,B]$ is the following condition for the principal symbol:

$$\{a,b\} \geq 0 \quad \text{when} \quad a = b = 0.$$

More precisely, we ask that $\{a,b\}(x,\xi) \geq 0$ for any $(x,\xi) \in T^*M$ for which $a(x,\xi) = b(x,\xi) = 0$. The idea is that in Lemma 3.3 one has the nonnegative terms $\|Au\|^2$ and $\|Bu\|^2$, and if either of these is large then it may cancel a negative contribution from the commutator term. On the level of symbols, one therefore only needs positivity of $\{a, b\}$ when the principal symbols of A and B vanish.

Recall that one wants the estimate (3.3) also for $P_{-\varphi}$. Changing φ to $-\varphi$ in Lemma 3.2, we see that $P_{-\varphi} = A - iB$. As in Lemma 3.3 one then asks a positivity condition for $i[A, -B]$, which has principal symbol $-\{a, b\}$. Thus, we also require that

$$\{a, b\} \leq 0 \quad \text{when} \quad a = b = 0.$$

Combining the above conditions for $\{a, b\}$, we have finally arrived at the definition of limiting Carleman weights. The definition is most naturally stated on open manifolds, and it includes the useful additional condition that φ should have nonvanishing gradient.

Definition. Let (M, g) be an open Riemannian manifold. We say that a smooth real valued function φ in M is a *limiting Carleman weight (LCW)* if $d\varphi \neq 0$ in M and

$$\{a, b\} = 0 \quad \text{when} \quad a = b = 0.$$

Example. Let $(M, g) = (\Omega, e)$ where Ω is an open set in $\mathbb{R}^n$. We will verify that the linear function $\varphi(x) = \alpha \cdot x$, where $\alpha \in \mathbb{R}^n$ is a nonzero vector, is an LCW. Indeed, one has $\nabla\varphi = \alpha$ and the principal symbols are

$$a(x, \xi) = |\xi|^2 - |\alpha|^2,$$
$$b(x, \xi) = 2\alpha \cdot \xi.$$

Since a and b are independent of x, the Poisson bracket is

$$\{a, b\} = \nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b \equiv 0.$$

Thus φ is an LCW.

Exercise 3.3. If $(M, g) = (\Omega, e)$ and $0 \notin \overline{\Omega}$, verify that $\varphi(x) = \log |x|$ and $\varphi(x) = \frac{\alpha \cdot x}{|x|^2}$ are LCWs. Here $\alpha \in \mathbb{R}^n$ is a fixed vector.

3.2. Characterization

In the previous section, after a long motivation we ended up with a definition of LCWs involving a rather abstract vanishing condition for a certain Poisson bracket. Here we give a geometric meaning to this condition, and also prove Theorem 3.1 which characterizes all Riemannian manifolds which admit LCWs. We recall the statement.

THEOREM 3.1. *(Manifolds which admit LCWs) Let (M, g) be a simply connected open Riemannian manifold. Then (M, g) admits an LCW iff some conformal multiple of g admits a parallel unit vector field.*

Recall that a vector field X is parallel if $D_V X = 0$ for any vector field V. Also recall that a manifold is simply connected if it is connected and if every closed curve is homotopic to a point. An explanation of the geometric condition, including examples of manifolds which satisfy it, is given in the next section.

We now begin the proof of Theorem 3.1. Let (M, g) be an open manifold. Recall that $\varphi \in C^\infty(M\,;\mathbb{R})$ is an LCW if $d\varphi \neq 0$ in M and

$$\{a, b\} = 0 \quad \text{when} \quad a = b = 0.$$

Here $a(x, \xi) = |\xi|^2 - |\nabla\varphi|^2$ and $b(x, \xi) = 2\langle d\varphi, \xi\rangle$ are smooth functions in T^*M. The first step is to find an expression for the Poisson bracket $\{a, b\}$, defined in local coordinates by $\{a, b\} := \nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b$.

MOTIVATION. We first compute the Poisson bracket in $\mathbb{R}^n$. Then $a(x, \xi) = |\xi|^2 - |\nabla\varphi|^2$ and $b(x, \xi) = 2\nabla\varphi \cdot \xi$, and writing φ'' for the Hessian matrix $(\varphi_{x_j x_k})_{j,k=1}^n$ we have

$$\begin{aligned}
\{a, b\} &= \nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b \\
&= 2\xi \cdot 2\varphi''\xi - (-2\varphi''\nabla\varphi) \cdot 2\nabla\varphi \\
&= 4\varphi''\xi \cdot \xi + 4\varphi''\nabla\varphi \cdot \nabla\varphi.
\end{aligned}$$

A computation in normal coordinates will show that a similar expression, now involving the covariant Hessian, holds on a Riemannian manifold.

LEMMA 3.5. *(Expression for Poisson bracket) The Poisson bracket is given by*

$$\{a, b\}(x, \xi) = 4D^2\varphi(\xi^\sharp, \xi^\sharp) + 4D^2\varphi(\nabla\varphi, \nabla\varphi).$$

PROOF. Both sides are invariantly defined functions on T^*M, so it is enough to check the identity in some local coordinates at a given point. Fix $p \in M$, let x be normal coordinates centered at p, and let (x, ξ) be the associated local coordinates in T^*M near p. Then

$$\begin{aligned}
a(x, \xi) &= g^{jk}\xi_j\xi_k - g^{jk}\varphi_{x_j}\varphi_{x_k}, \\
b(x, \xi) &= 2g^{jk}\varphi_{x_j}\xi_k.
\end{aligned}$$

Using that $g^{jk}|_p = \delta^{jk}$ and $\partial_l g^{jk}|_p = \Gamma^l_{jk}|_p = 0$, we have

$$
\begin{aligned}
\{a,b\}(x,\xi)|_p &= \sum_{l=1}^n \left[\partial_{\xi_l} a \partial_{x_l} b - \partial_{x_l} a \partial_{\xi_l} b\right]\Big|_p \\
&= \sum_{l=1}^n \left[(2g^{jl}\xi_l)(2g^{jk}\varphi_{x_j x_l}\xi_k) - (-2g^{jk}\varphi_{x_j x_l}\varphi_{x_k})(2g^{jl}\varphi_{x_j})\right]\Big|_p \\
&= \sum_{j,l=1}^n \left[4\varphi_{x_j x_l}\xi_j\xi_l + 4\varphi_{x_j x_l}\varphi_{x_j}\varphi_{x_l}\right]\Big|_p \\
&= (4D^2\varphi(\xi^\sharp,\xi^\sharp) + 4D^2\varphi(\nabla\varphi,\nabla\varphi))|_p
\end{aligned}
$$

since $D^2\varphi|_p = \varphi_{x_j x_l} dx^j \otimes dx^l|_p$. □

This immediately implies a condition for LCWs which is easier to work with than the original one.

Corollary 3.6. *φ is an LCW iff $d\varphi \neq 0$ in M and*

$$D^2\varphi(X,X) + D^2\varphi(\nabla\varphi,\nabla\varphi) = 0 \text{ when } |X| = |\nabla\varphi| \text{ and } \langle X,\nabla\varphi\rangle = 0.$$

We can now give a full characterization of LCWs in two dimensions. To do this, recall that the trace of a 2-tensor S on an n-dimensional manifold (N,g) is (analogously to the trace of an $n\times n$ matrix) defined by

$$\mathrm{Tr}(S)|_p := \sum_{j=1}^n S(e_j,e_j)$$

where $\{e_1,\dots,e_n\}$ is any orthonormal basis of T_pN. The trace of the Hessian is just the Laplace-Beltrami operator, as may be seen by a computation in normal coordinates at p:

$$\mathrm{Tr}(D^2\varphi)|_p = \sum_{j=1}^n D^2\varphi(\partial_j,\partial_j)|_p = \sum_{j=1}^n \varphi_{x_j x_j}|_p = \Delta_g\varphi|_p.$$

Proposition 3.7. *(LCWs in 2D) The LCWs in a 2D manifold (M,g) are exactly the harmonic functions with nonvanishing differential.*

Proof. If $|X| = |\nabla\varphi|$ and $\langle X,\nabla\varphi\rangle = 0$, then $\{X/|\nabla\varphi|, \nabla\varphi/|\nabla\varphi|\}$ is an orthonormal basis of the tangent space. Then

$$D^2\varphi(X,X) + D^2\varphi(\nabla\varphi,\nabla\varphi) = |\nabla\varphi|^2\mathrm{Tr}(D^2\varphi) = |\nabla\varphi|^2\Delta_g\varphi.$$

By Corollary 3.6, φ is an LCW iff $\Delta_g\varphi = 0$ and $d\varphi \neq 0$. □

After having characterized the situation in two dimensions, we move on to the case $n \geq 3$. The crucial fact here is that the existence of LCWs is a conformally invariant condition.

PROPOSITION 3.8. *(Existence of LCWs only depends on conformal class) If φ is an LCW in (M, g), then φ is an LCW in (M, cg) for any smooth positive function c.*

PROOF. Suppose φ is an LCW in (M, g), and let $\tilde{g} = cg$. Then the symbols $\tilde{a}$ and $\tilde{b}$ for the metric $\tilde{g}$ are

$$\begin{aligned}\tilde{a} &= \tilde{g}^{jk}\xi_j\xi_k - \tilde{g}^{jk}\varphi_{x_j}\varphi_{x_k} = c^{-1}(g^{jk}\xi_j\xi_k - g^{jk}\varphi_{x_j}\varphi_{x_k}) = c^{-1}a,\\ \tilde{b} &= 2\tilde{g}^{jk}\varphi_{x_j}\xi_k = 2c^{-1}g^{jk}\varphi_{x_j}\xi_k = c^{-1}b.\end{aligned}$$

Since c^{-1} does not depend on ξ, it follows that

$$\begin{aligned}\{\tilde{a}, \tilde{b}\} &= \{c^{-1}a, c^{-1}b\} = c^{-1}\nabla_\xi a \cdot \nabla_x(c^{-1}b) - c^{-1}\nabla_x(c^{-1}a) \cdot \nabla_\xi b\\ &= c^{-2}\{a, b\} + c^{-1}b\{a, c^{-1}\} + c^{-1}a\{c^{-1}, b\}.\end{aligned}$$

Suppose that $\tilde{a} = \tilde{b} = 0$. Then $a = b = 0$, and using that φ is an LCW it follows that $\{a, b\} = 0$. Consequently $\{\tilde{a}, \tilde{b}\} = 0$ when $\tilde{a} = \tilde{b} = 0$, showing that φ is an LCW in $(M, \tilde{g})$. $\square$

At this point we record a lemma which expresses relations between the Hessian and the covariant derivative.

LEMMA 3.9. *If $\varphi \in C^\infty(M)$ then*

$$\begin{aligned}D^2\varphi(X, Y) &= \langle D_X\nabla\varphi, Y\rangle,\\ D^2\varphi(X, \nabla\varphi) &= \langle D_X\nabla\varphi, \nabla\varphi\rangle = \frac{1}{2}X(|\nabla\varphi|^2),\\ D^2\varphi(\dot{\gamma}(t), \dot{\gamma}(t)) &= \frac{d^2}{dt^2}\varphi(\gamma(t))\end{aligned}$$

for any X, Y and for any geodesic γ.

PROOF. The first identity follows from a computation in normal coordinates. The second identity follows from the first one and the metric property of D. The third identity holds since

$$\begin{aligned}\frac{d^2}{dt^2}\varphi(\gamma(t)) &= \frac{d}{dt}\langle\nabla\varphi(\gamma(t)), \dot{\gamma}(t)\rangle = \langle D_{\dot{\gamma}(t)}\nabla\varphi(\gamma(t)), \dot{\gamma}(t)\rangle\\ &= D^2\varphi(\dot{\gamma}(t), \dot{\gamma}(t))\end{aligned}$$

by the first identity. Here we used that $D_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ since γ is a geodesic. $\square$

Using the second identity in the previous lemma, we now observe that if φ is an LCW and additionally $|\nabla\varphi| = 1$, then the second term in Corollary 3.6 vanishes:

$$D^2\varphi(\nabla\varphi, \nabla\varphi) = \frac{1}{2}\nabla\varphi(|\nabla\varphi|^2) = 0.$$

A smooth function which satisfies $|\nabla\varphi| = 1$ is called a *distance function* (since any such function is locally given by the Riemannian distance to a point or submanifold, but we will not need this fact). If one is given an LCW φ in (M, g), one can always reduce to the case where the LCW is a distance function by using the following conformal transformation.

LEMMA 3.10. *(Conformal normalization) If φ is a smooth function in (M, g) and if $\tilde{g} = |\nabla\varphi|^2 g$, then $|\nabla_{\tilde{g}}\varphi|_{\tilde{g}} = 1$.*

PROOF. $|\nabla_{\tilde{g}}\varphi|^2_{\tilde{g}} = \tilde{g}^{jk}\varphi_{x_j}\varphi_{x_k} = |\nabla\varphi|^{-2}g^{jk}\varphi_{x_j}\varphi_{x_k} = 1.$ □

We have an important characterization of LCWs which are also distance functions.

LEMMA 3.11. *(LCWs which are distance functions) Let $\varphi \in C^\infty(M)$ and $|\nabla\varphi| = 1$. The following conditions are equivalent:*

(1) *φ is an LCW.*
(2) $D^2\varphi \equiv 0$.
(3) *$\nabla\varphi$ is parallel.*
(4) *If $p \in M$ and if v is in the domain of $\exp_p$, then*

$$\varphi(\exp_p(v)) = \varphi(p) + \langle\nabla\varphi(p), v\rangle.$$

PROOF. Since $|\nabla\varphi| = 1$ we have $D^2\varphi(\nabla\varphi, \nabla\varphi) = 0$. Thus by Corollary 3.6, φ is an LCW iff

$$D^2\varphi(X, X) = 0 \text{ when } |X| = 1 \text{ and } \langle X, \nabla\varphi\rangle = 0.$$

Since $D^2\varphi$ is bilinear we may drop the condition $|X| = 1$, and the condition for LCW becomes

$$D^2\varphi(X, X) = 0 \text{ when } \langle X, \nabla\varphi\rangle = 0.$$

(1) $\implies$ (2): Suppose φ is an LCW. Fix $p \in M$ and choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM such that $e_1 = \nabla\varphi$. Then, by the above discussion,

$$\begin{aligned} &D^2\varphi(e_1, e_1) = 0, \\ &D^2\varphi(e_j, e_k) = 0 \quad \text{for } 2 \leq j, k \leq n. \end{aligned}$$

By Lemma 3.9 we also have $D^2\varphi(X, \nabla\varphi) = \frac{1}{2}X(|\nabla\varphi|^2) = 0$ for any X, therefore

$$D^2\varphi(e_j, e_1) = 0 \quad \text{for } 2 \leq j \leq n.$$

Since $D^2\varphi$ is bilinear and symmetric, we obtain $D^2\varphi \equiv 0$.

(2) $\implies$ (1): This is immediate.

(2) $\iff$ (3): Follows from $D^2\varphi(X, Y) = \langle D_X \nabla\varphi, Y\rangle$.

(2) $\iff$ (4): Let $\gamma_v(t) = \exp_p(tv)$. Then

$$\frac{d}{dt}\varphi(\gamma_v(t)) = \langle \nabla\varphi(\gamma_v(t)), \dot{\gamma}_v(t)\rangle,$$
$$\frac{d^2}{dt^2}\varphi(\gamma_v(t)) = D^2\varphi(\dot{\gamma}_v(t), \dot{\gamma}_v(t)).$$

If $D^2\varphi \equiv 0$ then the second derivative of $\varphi(\gamma_v(t))$ vanishes, therefore $\varphi(\gamma_v(t)) = a_0 + b_0 t$ for some real constants a_0, b_0. Evaluating $\varphi(\gamma_v(t))$ and its derivative at $t = 0$ gives

$$\varphi(\exp_p(tv)) = \varphi(p) + \langle \nabla\varphi(p), v\rangle t.$$

Conversely, if the last identity is valid then the second derivative of $\varphi(\gamma_v(t))$ vanishes, which implies $D^2\varphi \equiv 0$. □

Remarks. 1. The condition (4) indicates that LCWs which are also distance functions (normalized so that $\varphi(p) = 0$) are the analog on Riemannian manifolds of the linear Carleman weights in Euclidean space.

2. If φ is an LCW and a distance function, the above lemma shows that the Poisson bracket $\{a, b\}$ vanishes on all of T^*M instead of just on the submanifold where $a = b = 0$.

We have now established all the statements needed for the proof of Theorem 3.1, except for the fact that any parallel vector field in a simply connected manifold is a gradient field. Leaving this fact to the next section, we give the proof of the main theorem.

Proof of Theorem 3.1. Let (M, g) be simply connected and open.

" $\implies$ ": Suppose φ is an LCW in (M, g). By conformal invariance (Lemma 3.8) we know that φ is an LCW in $(M, \tilde{g})$ where $\tilde{g} = |\nabla\varphi|^2 g$. Lemma 3.10 shows that φ is also a distance function in $(M, \tilde{g})$. Then Lemma 3.11 applies, and we see that $\nabla_{\tilde{g}}\varphi$ is a unit parallel vector field in $(M, \tilde{g})$.

" $\Longleftarrow$ ": Assume that X is a unit parallel vector field in (M, cg) where $c > 0$. Since M is simply connected, the fact mentioned just before this proof shows that $X = \nabla_{cg}\varphi$ for some smooth function φ. Since $\nabla_{cg}\varphi$ is parallel and $|\nabla_{cg}\varphi|_{cg} = 1$, Lemma 3.11 implies that φ is an LCW in (M, cg). By conformal invariance φ is then an LCW also in (M, g). $\square$

3.3. Geometric interpretation

The geometric meaning of having a parallel unit vector field is given in the following result.

LEMMA 3.12. *(Parallel field $\Leftrightarrow$ product structure) Let X be a unit parallel vector field in (M, g). Near any point of M there exist local coordinates $x = (x_1, x')$ such that $X = \partial_1$ and*

$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \text{ for some metric } g_0 \text{ in the } x' \text{ variables.}$$

Conversely, if g is of this form then ∂_1 is a unit parallel vector field.

This says that the existence of a unit parallel vector field X implies that M is locally isometric to a subset of $(\mathbb{R}, e) \times (M_0, g_0)$ for some $(n-1)$-dimensional manifold (M_0, g_0). One can think of the direction of X as being a Euclidean direction on the manifold.

Note that any parallel vector field X has constant length on each component of M, since $V(|X|^2) = 2\langle D_V X, X\rangle = 0$ for any vector field V. Thus the existence of any nontrivial parallel vector field implies a product structure.

Theorem 3.1 now says that (M, g) admits an LCW iff up to a conformal factor there is a Euclidean direction on the manifold. More precisely:

LEMMA 3.13. *(LCWs in local coordinates) Let φ be an LCW in (M, g). Near any point of M there are local coordinates $x = (x_1, x')$ such that in these coordinates $\varphi(x) = x_1$ and*

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}$$

where c is a positive function and g_0 is some metric in the x' variables. Conversely, any metric of this form has the LCW $\varphi(x) = x_1$.

EXERCISE 3.4. Prove this lemma.

EXAMPLE. Manifolds which admit LCWs include the following:

1. Euclidean space $\mathbb{R}^n$ since any constant vector field is parallel,
2. all open subsets of the model spaces $\mathbb{R}^n$, $S^n \smallsetminus \{p_0\}$, and H^n since these are conformal to Euclidean space,
3. more general manifolds locally conformal to $\mathbb{R}^n$, such as symmetric spaces in 3D, admit LCWs locally,
4. all 2D manifolds admit LCWs at least locally by Proposition 3.7,
5. (Ω, g) admits an LCW if $\Omega \subseteq \mathbb{R}^n$ and if in some coordinates $x = (x_1, x')$ the metric g has the form
$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}$$
for some positive function c and some $(n-1)$-dimensional metric g_0.

The rest of this section is devoted to the proofs of Lemma 3.12 and the fact which was used in the proof of Theorem 3.1. We start with the latter.

LEMMA 3.14. *If M is a manifold with $H^1_{\mathrm{dR}}(M) = \{0\}$, then any parallel unit vector field on M is a gradient field.*

PROOF. Let X be a parallel unit vector field on M. We choose $\omega = X^\flat$ to be the 1-form corresponding to X. It is enough to prove that $d\omega = 0$, since then the condition on the first de Rham cohomology group implies that $\omega = d\varphi$ for some smooth function φ and consequently $X = (d\varphi)^\sharp = \nabla\varphi$.

The fact that $d\omega = 0$ follows from the general identity
$$d(X^\flat)(Y, Z) = \langle D_Y X, Z\rangle - \langle D_Z X, Y\rangle$$
since $D_V X = 0$ for any V. $\square$

EXERCISE 3.5. Show the identity used in the proof.

To prove Lemma 3.12 we need a version of the Frobenius theorem. For this purpose we introduce some terminology, see [**11**, Section 14] for more details. A *k-plane field* on a manifold M is a rule Γ which associates to each point p in M a k-dimensional subspace Γ_p of T_pM, such that Γ_p varies smoothly with p. A vector field X on M is called a *section of* Γ if $X(p) \in \Gamma_p$ for any p. A k-plane field Γ is called *involutive* if for any V, W which are sections of Γ, also the Lie bracket $[V, W]$ is a section of Γ.

THEOREM. *(Frobenius) If Γ is an involutive k-plane field, then through any point p in M there is an integral manifold S of Γ (that is, S is a k-dimensional submanifold of M with $\Gamma|_S = TS$).*

The other tool that is needed is a special local coordinate system called *semigeodesic coordinates.* The usual geodesic normal coordinates are obtained by following geodesic rays starting at a given point. Semi-geodesic coordinates are instead obtained by following geodesics which are normal to a given hypersurface S. On manifolds with boundary, semigeodesic coordinates where S is part of the boundary are called *boundary normal coordinates.*

LEMMA 3.15. *(Semigeodesic coordinates) Let $p \in M$ and let S be a hypersurface through p. There is a chart (U, x) at p such that $S \cap U = \{x_1 = 0\}$, the curves $x_1 \mapsto (x_1, x')$ correspond to normal geodesics starting from S, and the metric has the form*

$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x_1, x') \end{pmatrix}.$$

The inverse of the map $(x_1, x') \mapsto \exp_{q(x')}(x_1 N(q(x')))$ gives such a chart, where $x' \mapsto q(x')$ is a parametrization of S near p and N is a unit normal vector field of S.

EXERCISE 3.6. Prove this lemma.

PROOF OF LEMMA 3.12. " $\Longrightarrow$ " Let X be unit parallel, and let Γ be the $(n-1)$-plane field consisting of vectors orthogonal to X. If V, W are vector fields orthogonal to X then

$$\langle [V, W], X \rangle = \langle D_V W - D_W V, X \rangle = V \langle W, X \rangle - W \langle V, X \rangle = 0$$

using the symmetry and metric property of the Levi-Civita connection and the fact that X is parallel. This shows that Γ is an involutive $(n-1)$-plane field.

Fix $p \in M$, and use the Frobenius theorem to find a hypersurface S through p such that X is normal to S. If $x' \mapsto q(x')$ parametrizes S near p, then $(x_1, x') \mapsto \exp_{q(x')}(x_1 X(q(x')))$ gives semigeodesic coordinates near p such that ∂_1 is the tangent vector of a normal geodesic to S and

$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x_1, x') \end{pmatrix}.$$

Now the integral curves of X are geodesics (if $\dot\gamma(t) = X(\gamma(t))$ then $D_{\dot\gamma(t)}\dot\gamma(t) = D_{\dot\gamma(t)}X(\gamma(t)) = 0$), which shows that $X = \partial_1$. It remains to prove that $g_0(x_1, x')$ is independent of x_1. But for $j, k \geq 2$ we have

$$\begin{aligned}\partial_1 g_{jk} &= \partial_1 \langle \partial_j, \partial_k \rangle = \langle D_{\partial_1}\partial_j, \partial_k \rangle + \langle \partial_j, D_{\partial_1}\partial_k \rangle \\ &= \langle D_{\partial_j}\partial_1, \partial_k \rangle + \langle \partial_j, D_{\partial_k}\partial_1 \rangle = 0\end{aligned}$$

since $D_{\partial_1}\partial_l - D_{\partial_l}\partial_1 = [\partial_1, \partial_l] = 0$ and since $\partial_1 = X$ is parallel.

" $\Longleftarrow$ " Exercise. □

EXERCISE 3.7. Prove the converse direction in Lemma 3.12.

CHAPTER 4

Carleman estimates

In the previous chapter we introduced limiting Carleman weights (LCWs), motivated by the possibility of constructing special solutions to the Schrödinger equation $(-\Delta_g + V)u = 0$ in M having the form

$$u = e^{\pm\varphi/h}(a + r)$$

where φ is an LCW, $h > 0$ is a small parameter, and the correction term r converges to zero as $h \to 0$. The arguments involved solving inhomogeneous equations of the type

$$e^{\pm\varphi/h}(-\Delta_g + V)e^{\mp\varphi/h}r = f \quad \text{in } M \tag{4.1}$$

with the norm estimate

$$\|r\|_{L^2(M)} \leq Ch\|f\|_{L^2(M)}, \quad 0 < h < h_0.$$

We then gave a definition of LCWs based on an abstract condition on the vanishing of a Poisson bracket and proved that on a simply connected open manifold (M, g), by Theorem 3.1 and Lemma 3.13,

$$\begin{aligned} &\varphi \text{ is an LCW in } (M,g) \\ \iff\ &\nabla_{\tilde{c}g}\varphi \text{ is unit parallel in } (M,\tilde{c}g) \text{ for some } \tilde{c} > 0 \\ \implies\ &\text{locally in some coordinates } \varphi(x) = x_1 \text{ and } g = c(e \oplus g_0). \end{aligned}$$

On the last line, the notation means that $c^{-1}g$ is the product of the Euclidean metric e on $\mathbb{R}$ and some $(n-1)$-dimensional metric g_0.

In this chapter we will show that the existence of an LCW indeed implies the solvability of the inhomogeneous equation (4.1) with the right norm estimates. We will prove this under the extra assumption that the metric has the product structure $g = c(e \oplus g_0)$ *globally* instead of just locally. Following [**10**], this assumption makes it possible to use Fourier analysis to write down the solutions in a rather explicit way. See [**4**, Section 4] for a different (though less explicit) proof based on integration by parts arguments as in Section 3.1, which does not require the extra assumption on global structure of g.

4.1. Motivation and main theorem

As usual, we will first consider solvability of the inhomogeneous equation in the Euclidean case. Here and below we will consider a large parameter $\tau = 1/h$ instead of a small parameter. This is just a matter of notation, and this choice will be slightly more transparent (also the Fourier analysis proof will allow us to avoid semiclassical symbol calculus for which a small parameter would be more natural).

Motivation. Consider the analog of the equation (4.1) in $\mathbb{R}^n$ with the LCW $\varphi(x) = x_1$ and with $V = 0$,

$$e^{\tau x_1}(-\Delta)e^{-\tau x_1}u = f \quad \text{in } \mathbb{R}^n.$$

Noting that $e^{\tau x_1} D e^{-\tau x_1} = D + i\tau e_1$ where $D = -i\nabla$, we compute $e^{\tau x_1}(-\Delta)e^{-\tau x_1} = (D+i\tau e_1)^2 = -\Delta + 2\tau\partial_1 - \tau^2$. The equation becomes

$$(-\Delta + 2\tau\partial_1 - \tau^2)u = f \quad \text{in } \mathbb{R}^n.$$

The operator on the left has constant coefficients, and one can try to find a solution by taking the Fourier transform of both sides. Since $(D_j u)\hat{}(\xi) = \xi_j \hat{u}(\xi)$, this gives the equation

$$(|\xi|^2 + 2i\tau\xi_1 - \tau^2)\hat{u}(\xi) = \hat{f}(\xi) \quad \text{in } \mathbb{R}^n.$$

Thus, one formally obtains the solution

$$u = \mathscr{F}^{-1}\left\{ \frac{1}{p(\xi)} \hat{f}(\xi) \right\}$$

where $p(\xi) := |\xi|^2 - \tau^2 + 2i\tau\xi_1$. The problem is that the symbol $p(\xi)$ has zeros, and it is not immediately obvious if one can divide by $p(\xi)$. In fact the zero set of the symbol is a codimension 2 manifold,

$$p^{-1}(0) = \{\xi \in \mathbb{R}^n \,;\, |\xi| = |\tau|,\ \xi_1 = 0\}.$$

It was shown in [**19**] after a careful analysis that one can indeed justify the division by $p(\xi)$ if the functions are in certain weighted L^2 spaces. Define for $\delta \in \mathbb{R}$ the space

$$L^2_\delta(\mathbb{R}^n) := \{f \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \,;\, (1+|x|^2)^{\delta/2} f \in L^2(\mathbb{R}^n)\}.$$

The result of [**19**] states that if $-1 < \delta < 0$, then for any $f \in L^2_{\delta+1}(\mathbb{R}^n)$ this argument gives a unique solution $u \in L^2_\delta(\mathbb{R}^n)$ with the right norm estimates.

It turns out that a similar Fourier analysis argument will also work in the Riemannian case if the metric is related to the product metric on $\mathbb{R} \times M_0$. One can then use the ordinary Fourier transform on $\mathbb{R}$, but on the transversal manifold M_0 the Fourier transform is replaced by eigenfunction expansions. Also, since the spectrum in the transversal directions is discrete, it turns out we can easily avoid the problem of dividing by zero just by imposing a harmless extra condition on the large parameter τ.

In this chapter we will be working in a cylinder $T := \mathbb{R} \times M_0$ with metric $g := c(e \oplus g_0)$, where (M_0, g_0) is a compact $(n-1)$-dimensional manifold with boundary and $c > 0$ is a smooth positive function. We will write points of T as (x_1, x') where x_1 is the Euclidean coordinate on $\mathbb{R}$ and x' are local coordinates on M_0. Thus g has the form

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}.$$

Note that these coordinates and the representation of the metric are valid globally in x_1 and locally in M_0.

We denote by $L^2(T) = L^2(T, dV_g)$ the natural L^2 space on (T, g). The local L^2 space is

$$L^2_{\mathrm{loc}}(T) := \{f \,;\, f \in L^2([-R, R] \times M_0) \text{ for all } R > 0\}.$$

Writing $\langle x \rangle = (1 + |x|^2)^{1/2}$, we define for any $\delta \in \mathbb{R}$ the polynomially weighted (in the x_1 variable) spaces

$$\begin{aligned} L^2_\delta(T) &:= \{f \in L^2_{\mathrm{loc}}(T) \,;\, \langle x_1 \rangle^\delta f \in L^2(T)\}, \\ H^1_\delta(T) &:= \{f \in L^2_\delta(T) \,;\, df \in L^2_\delta(T)\}, \\ H^1_{\delta,0}(T) &:= \{f \in H^1_\delta(T) \,;\, f|_{\mathbb{R} \times M_0} = 0\}. \end{aligned}$$

These have natural norms

$$\begin{aligned} \|f\|_{L^2_\delta(T)} &:= \|\langle x_1 \rangle^\delta f\|_{L^2(T)}, \\ \|f\|_{H^1_\delta(T)} &:= \|\langle x_1 \rangle^\delta f\|_{L^2(T)} + \|\langle x_1 \rangle^\delta df\|_{L^2(T)}. \end{aligned}$$

More precisely, $L^2_\delta(T)$ and $H^1_\delta(T)$ are the completions in the respective norms of the space $\{f \in C^\infty(T) \,;\, f(x_1, x') = 0 \text{ for } |x_1| \text{ large}\}$, and $H^1_{\delta,0}(T)$ is the completion of $C^\infty_c(T^{\mathrm{int}})$ in the $H^1_\delta(T)$ norm.

If g has the special form given above, $\varphi(x) = x_1$ is a natural LCW. We denote by Δ_g and Δ_{g_0} the Laplace-Beltrami operators in (T, g) and (M_0, g_0), respectively. The main result is as follows.

THEOREM 4.1. *(Solvability and norm estimates) Let $\delta > 1/2$, assume that $c(x_1, x') = 1$ for $|x_1|$ large, and let V be a complex function in T with $\langle x_1 \rangle^{2\delta} V \in L^\infty(T)$. There exist $C_0, \tau_0 > 0$ such that whenever*

$$|\tau| \geq \tau_0 \quad \text{and} \quad \tau^2 \notin \mathrm{Spec}(-\Delta_{g_0}),$$

then for any $f \in L^2_\delta(T)$ there is a unique solution $u \in H^1_{-\delta,0}(T)$ of the equation

$$e^{\tau x_1}(-\Delta_g + V)e^{-\tau x_1} u = f \quad \text{in } T.$$

This solution satisfies

$$\|u\|_{L^2_{-\delta}(T)} \leq \frac{C_0}{|\tau|} \|f\|_{L^2_\delta(T)},$$
$$\|u\|_{H^1_{-\delta}(T)} \leq C_0 \|f\|_{L^2_\delta(T)}.$$

Here $\mathrm{Spec}(-\Delta_{g_0})$ is the discrete set of Dirichlet eigenvalues of $-\Delta_{g_0}$ in (M_0, g_0). The extra restriction $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$ allows us to avoid the problem of dividing by zero. One can always find a sequence of τ's converging to plus or minus infinity which satisfies this restriction, which is all that we will need for the applications to inverse problems. Typically, if we consider an inverse problem in a compact manifold (M, g) with boundary, Theorem 4.1 will be used by embedding (M, g) in a cylinder (T, g) of the above type and then solving the inhomogeneous equations in the larger manifold (T, g).

Let us formulate some open questions related to the above theorem (some of these questions should be quite doable).

QUESTION 4.1. *Prove an analog of Theorem 4.1 without the restriction $\tau^2 \notin$ Spec$(-\Delta_{g_0})$ by using slightly different function spaces.*

QUESTION 4.2. *(Existence of LCW implies global product structure) Find conditions on a manifold (M, g) such that the existence of an LCW on (M, g) would imply that $(M, g) \subset\subset (T, g)$ for a cylinder as above.*

QUESTION 4.3. *(Operators with first order terms) Prove an analog of Theorem 4.1 when the operator $-\Delta_g + V$ is replaced by $-\Delta_g + 2X + V$ where X is a vector field on T with suitable regularity and decay.*

4.2. Proof of the estimates

We begin the proof of Theorem 4.1. The first step is to observe that it is enough to prove the result for $c \equiv 1$. Note that the metric in T is of the form $c\tilde{g}$ where $\tilde{g} = e \oplus g_0$ is a product metric.

LEMMA 4.2. *(Schrödinger equation under conformal scaling) If c is a positive function in $(M,\tilde{g})$ and V is a function in M then*

$$c^{\frac{n+2}{4}}(-\Delta_{c\tilde{g}}+V)(c^{-\frac{n-2}{4}}v)=(-\Delta_{\tilde{g}}+[cV-c^{\frac{n+2}{4}}\Delta_g(c^{-\frac{n-2}{4}})])v.$$

EXERCISE 4.1. Prove the lemma.

Suppose now that Theorem 4.1 has been proved for the metric $\tilde{g}=e\oplus g_0$. For the general case $g=c\tilde{g}$, we need to produce a solution of

$$e^{\tau x_1}(-\Delta_{c\tilde{g}}+V)e^{-\tau x_1}u=f \quad \text{in } T.$$

We try $u=c^{-\frac{n-2}{4}}v$ for some v. By Lemma 4.2, it is enough to solve

$$e^{\tau x_1}(-\Delta_{\tilde{g}}+[cV-c^{\frac{n+2}{4}}\Delta_g(c^{-\frac{n-2}{4}})])e^{-\tau x_1}v=c^{\frac{n+2}{4}}f \quad \text{in } T.$$

But since $c=1$ for $|x_1|$ large, the potential $\tilde{V}:=cV-c^{\frac{n+2}{4}}\Delta_g(c^{-\frac{n-2}{4}})$ has the same decay properties as V (that is, $\tilde{V}\in\langle x_1\rangle^{2\delta}L^\infty(T)$). The right hand side $\tilde{f}:=c^{\frac{n+2}{4}}f$ is also in $L^2_\delta(T)$ like f, so Theorem 4.1 for $\tilde{g}$ implies the existence of a unique solution v. Since $u=c^{-\frac{n-2}{4}}v$ the solution u belongs to the same function spaces and satisfies similar estimates as v, and Theorem 4.1 follows in full generality.

From now on we will assume that $c\equiv 1$ and that we are working in (T,g) where $g=e\oplus g_0$, or in local coordinates

$$g(x_1,x')=\begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}.$$

Since $|g|$ only depends on x', the Laplace-Beltrami operator splits as

$$\Delta_g=\partial_1^2+\Delta_{g_0}.$$

Similarly, using that $e^{\tau x_1}D_1e^{-\tau x_1}=D_1+i\tau$, the conjugated Laplace-Beltrami operator has the expression

$$\begin{aligned} e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} &= (D_1+i\tau)^2-\Delta_{g_0} \\ &= -\partial_1^2+2\tau\partial_1-\tau^2-\Delta_{g_0}. \end{aligned}$$

Assuming that $V=0$ for the moment, the equation that we need to solve has now the form

$$(-\partial_1^2+2\tau\partial_1-\tau^2-\Delta_{g_0})u=f \quad \text{in } T. \tag{4.2}$$

As mentioned above, we will employ eigenfunction expansions in the manifold M_0 to solve the equation. Let $0<\lambda_1\leq\lambda_2\leq\ldots$ be the Dirichlet eigenvalues of the Laplace-Beltrami operator $-\Delta_{g_0}$ in

(M_0, g_0), and let ϕ_l be the corresponding Dirichlet eigenfunctions so that

$$-\Delta_{g_0}\phi_l = \lambda_l \phi_l \ \text{ in } M, \quad \phi_l \in H_0^1(M_0).$$

We assume that $\{\phi_l\}_{l=1}^\infty$ is an orthonormal basis of $L^2(M_0)$. Then, if f is a function on T such $f(x_1, \,\cdot\,) \in L^2(M_0)$ for almost every x_1, we define the partial Fourier coefficients

$$\tilde{f}(x_1, l) := \int_{M_0} f(x_1, x')\phi_l(x')\, dV_{g_0}(x'). \tag{4.3}$$

One has the eigenfunction expansion

$$f(x_1, x') = \sum_{l=1}^\infty \tilde{f}(x_1, l)\phi_l(x')$$

with convergence in $L^2(M_0)$ for almost every x_1.

Motivation. Formally, the proof of Theorem 4.1 now proceeds as follows. We consider eigenfunction expansions

$$u(x_1, x') = \sum_{l=1}^\infty \tilde{u}(x_1, l)\phi_l(x'), \quad f(x_1, x') = \sum_{l=1}^\infty \tilde{f}(x_1, l)\phi_l(x').$$

Inserting these expansions in (4.2) and using that $-\Delta_{g_0}\phi_l = \lambda_l\phi_l$ results in the following ODEs for the partial Fourier coefficients:

$$(-\partial_1^2 + 2\tau\partial_1 - \tau^2 + \lambda_l)\tilde{u}(\,\cdot, l) = \tilde{f}(\,\cdot\,, l) \quad \text{for all } l. \tag{4.4}$$

The easiest way to prove uniqueness of solutions is to take Fourier transforms in the x_1 variable. If the ODEs (4.4) are satisfied with zero right hand side, then with obvious notations

$$(\xi_1^2 + 2i\tau\xi_1 - \tau^2 + \lambda_l)\hat{u}(\xi_1, l) = 0 \quad \text{for all } l.$$

Now if the symbol $p(\xi_1, l) := \xi_1^2 + 2i\tau\xi_1 - \tau^2 + \lambda_l$ would be zero, looking at real and imaginary parts would imply $\xi_1 = 0$ and $\tau^2 = \lambda_l$. But the condition $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$ shows that this is not possible. Thus $p(\xi_1, l)$ is nonvanishing, and we obtain $\hat{u}(\xi_1, l) \equiv 0$ and consequently $u \equiv 0$. This proves uniqueness.

To show existence with the right norm estimates we observe that $-\partial_1^2 + 2\tau\partial_1 - \tau^2 = -(\partial_1 - \tau)^2$, and we factor (4.4) as

$$(\partial_1 - \tau - \sqrt{\lambda_l})(\partial_1 - \tau + \sqrt{\lambda_l})\tilde{u}(\,\cdot\,, l) = -\tilde{f}(\,\cdot\,, l) \quad \text{for all } l.$$

The Fourier coefficients of the solution u are then obtained from the Fourier coefficients of f by solving two ODEs of first order.

After this formal discussion, we will give the rigorous arguments which lie behind these ideas. Let us begin with uniqueness.

PROPOSITION 4.3. *(Uniqueness for $V = 0$) Let $u \in H^1_{\delta,0}(T)$ for some $\delta \in \mathbb{R}$, let $\tau^2 \notin Spec(-\Delta_{g_0})$, and assume that u satisfies*

$$(-\partial_1^2 + 2\tau\partial_1 - \tau^2 - \Delta_{g_0})u = 0 \quad \textit{in } T.$$

Then $u = 0$.

PROOF. The condition that u is a solution implies that

$$\int_T u(-\partial_1^2 - 2\tau\partial_1 - \tau^2 - \Delta_{g_0})\psi \, dV_g = 0$$

for any $\psi \in C_c^\infty(T^{\mathrm{int}})$. We make the choice $\psi(x_1, x') = \chi(x_1)\phi_{lj}(x')$ where $\chi \in C_c^\infty(\mathbb{R})$ and $\phi_{lj} \in C_c^\infty(M_0^{\mathrm{int}})$ with $\phi_{lj} \to \phi_l$ in $H^1(M_0)$ as $j \to \infty$. The last fact is possible since $\phi_l \in H_0^1(M_0)$. Now $g = e \oplus g_0$, so we have for any w

$$\int_T w \, dV_g = \int_{-\infty}^\infty \int_{M_0} w(x_1, x') \, dV_{g_0}(x') \, dx_1.$$

Thus, with this choice of ψ we obtain that

$$\begin{aligned} \text{(4.5)} \quad & \int_{-\infty}^\infty \left(\int_{M_0} u(x_1, \,\cdot\,)\phi_{lj} \, dV_{g_0} \right) (-\partial_1^2 - 2\tau\partial_1 - \tau^2)\chi(x_1) \, dx_1 \\ & \qquad + \int_{-\infty}^\infty \left(\int_{M_0} u(x_1, \,\cdot\,)(-\Delta_{g_0}\phi_{lj}) \, dV_{g_0} \right) \chi(x_1) \, dx_1 = 0. \end{aligned}$$

Note that $u(x_1, \,\cdot\,) \in H_0^1(M_0)$ for almost every x_1, because of the assumption $u \in H^1_{\delta,0}(T)$ and the facts

$$\int_{-\infty}^\infty \langle x_1 \rangle^{2\delta} \|u(x_1, \,\cdot\,)\|^2_{L^2(M_0)} \, dx_1 = \|u\|_{L^2_\delta(T)} < \infty,$$

$$\int_{-\infty}^\infty \langle x_1 \rangle^{2\delta} \|\nabla_{g_0} u(x_1, \,\cdot\,)\|^2_{L^2(M_0)} \, dx_1 = \|\nabla_{g_0} u\|_{L^2_\delta(T)} < \infty.$$

Since $-\Delta_{g_0}$ is an isomorphism $H_0^1(M_0) \to H^{-1}(M_0)$, we have

$$\int_{M_0} u(x_1, \,\cdot\,)\phi_{lj} \, dV_{g_0} \to \tilde{u}(x_1, l),$$

$$\int_{M_0} u(x_1, \,\cdot\,)(-\Delta_{g_0}\phi_{lj}) \, dV_{g_0} \to \lambda_l \tilde{u}(x_1, l)$$

as $j \to \infty$ for any x_1 such that $u(x_1, \,\cdot\,) \in H_0^1(M_0)$. Dominated convergence shows that we may take the limit in (4.5) and obtain

$$\int_{-\infty}^{\infty} \tilde{u}(x_1, l)(-\partial_1^2 - 2\tau\partial_1 - \tau^2 + \lambda_l)\chi(x_1)\, dx_1 = 0 \quad \text{for all } l.$$

The condition $u \in L^2_\delta(T)$ ensures that $\tilde{u}(\,\cdot\,, l) \in \langle\,\cdot\,\rangle^{-\delta} L^2(\mathbb{R})$, and the last identity implies

$$(-\partial_1^2 + 2\tau\partial_1 - \tau^2 + \lambda_l)\tilde{u}(\,\cdot\,, l) = 0 \quad \text{for all } l.$$

It only remains to take the Fourier transform in x_1 (which can be done in the sense of tempered distributions on $\mathbb{R}$), which gives

$$(\xi_1^2 + 2i\tau\xi_1 - \tau^2 + \lambda_l)\hat{u}(\,\cdot\,, l) = 0 \quad \text{for all } l.$$

The symbol $\xi_1^2 + 2i\tau\xi_1 - \tau^2 + \lambda_l$ is never zero because $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$. Thus $\tilde{u}(\,\cdot\,, l) = 0$ for all l, showing that $u(x_1, \,\cdot\,) = 0$ for almost every x_1 and consequently $u = 0$. $\square$

As discussed above, the existence of solutions will be established via certain first order ODEs. The next result gives the required solvability results and norm estimates. Here $L^2_\delta(\mathbb{R})$ is the space defined via the norm $\|f\|_{L^2_\delta(\mathbb{R})} := \|\langle x\rangle^\delta f\|_{L^2(\mathbb{R})}$, and $\mathscr{S}'(\mathbb{R})$ is the space of tempered distributions in $\mathbb{R}$.

Proposition 4.4. *(Solvability and norm estimates for an ODE) Let a be a nonzero real number, and consider the equation*

$$u' - au = f \quad \textit{in } \mathbb{R}.$$

For any $f \in \mathscr{S}'(\mathbb{R})$ there is a unique solution $u \in \mathscr{S}'(\mathbb{R})$. Writing $S_a f := u$, we have the mapping properties

$$\begin{aligned} S_a &: L^2_\delta(\mathbb{R}) \to L^2_\delta(\mathbb{R}) \quad \textit{for all } \delta \in \mathbb{R}, \\ S_a &: L^1(\mathbb{R}) \to L^\infty(\mathbb{R}), \end{aligned}$$

and the norm estimates

$$\begin{aligned} \|S_a f\|_{L^2_\delta} &\le \frac{C_\delta}{|a|}\|f\|_{L^2_\delta} \quad \textit{if } |a| \ge 1 \textit{ and } \delta \in \mathbb{R}, \\ \|S_a f\|_{L^2_{-\delta}} &\le C_\delta \|f\|_{L^2_\delta} \quad \textit{if } a \ne 0 \textit{ and } \delta > 1/2, \\ \|S_a f\|_{L^\infty} &\le \|f\|_{L^1}. \end{aligned}$$

Proof. *Step 1.* Let us first consider solvability in $\mathscr{S}'(\mathbb{R})$. Taking Fourier transforms, we have

$$\begin{aligned} u' - au = f &\iff (i\xi - a)\hat{u} = \hat{f} \\ &\iff u = \mathscr{F}^{-1}\{m(\xi)\hat{f}(\xi)\} \end{aligned}$$

with $m(\xi) := (i\xi - a)^{-1}$. Since $a \neq 0$ the function m is smooth and its derivatives are given by $m^{(k)}(\xi) = (-i)^k k!(i\xi - a)^{-k-1}$. Therefore

$$\|m^{(k)}\|_{L^\infty} \leq k!|a|^{-k-1}, \quad k = 0, 1, 2, \ldots. \tag{4.6}$$

Thus m has bounded derivatives and $v \mapsto mv$ is continuous on $\mathscr{S}'(\mathbb{R})$. It follows that $S_a f := \mathscr{F}^{-1}\{m(\xi)\hat{f}(\xi)\}$ produces for any $f \in \mathscr{S}'(\mathbb{R})$ a unique solution in $\mathscr{S}'(\mathbb{R})$ to the given ODE.

Step 2. Let $f \in L^2_\delta(\mathbb{R})$ where $\delta \in \mathbb{R}$. We will use the following Sobolev space facts on $\mathbb{R}$: if $\|m\|_{W^{k,\infty}} := \sum_{j=0}^k \|m^{(j)}\|_{L^\infty}$ then

$$\|v\|_{H^\delta} = \|\langle\,\cdot\,\rangle^\delta \hat{v}\|_{L^2} = \|\hat{v}\|_{L^2_\delta}, \tag{4.7}$$

$$\|mv\|_{H^\delta} \leq C_\delta \|m\|_{W^{k,\infty}} \|v\|_{H^\delta} \quad \text{when } k \geq |\delta|. \tag{4.8}$$

Then for $k \geq |\delta|$

$$\begin{aligned} \|S_a f\|_{L^2_\delta} &= \|\mathscr{F}^{-1}\{S_a f\}\|_{H^\delta} = (2\pi)^{-1}\|(m\hat{f})(-\,\cdot\,)\|_{H^\delta} \\ &\leq C_\delta (2\pi)^{-1} \|m\|_{W^{k,\infty}} \|\hat{f}(-\,\cdot\,)\|_{H^\delta} \\ &= C_\delta \|m\|_{W^{k,\infty}} \|f\|_{L^2_\delta}. \end{aligned}$$

This proves that S_a maps L^2_δ to itself. If additionally $|a| \geq 1$, the estimates (4.6) imply

$$\|S_a f\|_{L^2_\delta} \leq \frac{C_\delta}{|a|} \|f\|_{L^2_\delta}.$$

Step 3. Let $f \in L^1(\mathbb{R})$, and let $a > 0$ (the case $a < 0$ is analogous). To prove the $L^1 \to L^\infty$ bounds we will work on the spatial side and solve the ODE by using the standard method of integrating factors. In the sense of distributions

$$\begin{aligned} u' - au = f &\iff u'e^{-at} - aue^{-at} = fe^{-at} \\ &\iff (ue^{-at})' = fe^{-at}. \end{aligned}$$

Integrating both sides from x to ∞ (here we use that $a > 0$ so e^{-at} is decreasing as $t \to \infty$), we define

$$u(x) := -\int_x^\infty f(t)e^{-a(t-x)}\,dt.$$

Since $|e^{-a(t-x)}| \leq 1$ for $t \geq x$, uniformly over $a > 0$, we see that $\|u\|_{L^\infty} \leq \|f\|_{L^1}$. Since u clearly solves the ODE we have $u = S_a f$ by uniqueness of solutions. This shows the mapping property and norm estimates of S_a on L^1.

Step 4. Finally, let $f \in L^2_\delta(\mathbb{R})$ with $\delta > 1/2$. It remains to convert the $L^1 \to L^\infty$ estimate to a weighted L^2 estimate. Using that

$$c_\delta := \left(\int_{-\infty}^\infty \langle t\rangle^{-2\delta}\right)^{1/2} < \infty$$

for $\delta > 1/2$, we have

$$\begin{aligned}
\|S_a f\|_{L^2_{-\delta}} &= \left(\int_{-\infty}^\infty \langle t\rangle^{-2\delta}|S_a f(t)|^2\,dt\right)^{1/2} \\
&\leq c_\delta \|S_a f\|_{L^\infty} \\
&\leq c_\delta \|f\|_{L^1} \\
&= c_\delta \int_{-\infty}^\infty \langle t\rangle^{-\delta}\langle t\rangle^{\delta}|f(t)|\,dt \\
&\leq c_\delta^2 \|f\|_{L^2_\delta}.
\end{aligned}$$

The last inequality follows by Cauchy-Schwarz. $\square$

Exercise 4.2. Verify the Sobolev space facts (4.7), (4.8).

Remark 4.5. We will employ the $L^2_\delta \to L^2_\delta$ estimate when $|a| \geq 1$. The proof shows that when a is small then the constant in this estimate blows up. This is why we need the $L^2_\delta \to L^2_{-\delta}$ estimate for $\delta > 1/2$, with constant independent of a. The method for converting an $L^1 \to L^\infty$ estimate to a weighted L^2 estimate arises in Agmon's scattering theory for short range potentials. The weighted L^2 estimate is more convenient for our purposes than the stronger $L^1 \to L^\infty$ estimate since the weighted L^2 spaces will make it possible to use orthogonality.

We can now show the existence of solutions to the inhomogeneous equation with no potential.

PROPOSITION 4.6. *(Existence for $V = 0$) Let $f \in L^2_\delta(T)$ where $\delta > 1/2$. There is $C_0 > 0$ such that whenever $|\tau| \geq 1$ and $\tau^2 \notin Spec(-\Delta_{g_0})$, then the equation*

$$(-\partial_1^2 + 2\tau\partial_1 - \tau^2 - \Delta_{g_0})u = f \quad \text{in } T \tag{4.9}$$

has a solution $u \in H^1_{-\delta,0}(T)$ satisfying

$$\|u\|_{L^2_{-\delta}(T)} \leq \frac{C_0}{|\tau|}\|f\|_{L^2_\delta(T)},$$
$$\|u\|_{H^1_{-\delta}(T)} \leq C_0\|f\|_{L^2_\delta(T)}.$$

PROOF. *Step 1.* We begin with a remark on orthogonality. Since $f \in L^2_\delta(T)$, we know that $f(x_1, \,\cdot\,) \in L^2(M_0)$ for almost every x_1. Then for such x_1 the Parseval identity implies

$$\int_{L^2(M_0)} |f(x_1, x')|^2 \, dV_{g_0}(x') = \sum_{l=1}^{\infty} |\tilde{f}(x_1, l)|^2.$$

Here $\tilde{f}(x_1, l)$ are the Fourier coefficients (4.3). It follows that

$$\begin{aligned}
\|f\|^2_{L^2_\delta(T)} &= \int_{-\infty}^{\infty} \langle x_1 \rangle^{2\delta} \left(\int_{M_0} |f(x_1, x')|^2 \, dV_{g_0}(x') \right) dx_1 \\
&= \int_{-\infty}^{\infty} \langle x_1 \rangle^{2\delta} \left(\sum_{l=1}^{\infty} |\tilde{f}(x_1, l)|^2 \right) dx_1 \\
&= \sum_{l=1}^{\infty} \|\tilde{f}(\,\cdot\,, l)\|^2_{L^2_\delta(\mathbb{R})}.
\end{aligned}$$

In the last equality, we used Fubini's theorem which is valid since the integrand is nonnegative. In particular, this argument shows that $\tilde{f}(\,\cdot\,, l) \in L^2_\delta(\mathbb{R})$ for all l, and that the last sum converges.

Step 2. From now on we assume that $\tau > 0$ (the case $\tau < 0$ is analogous). Motivated by the discussion before (4.4), we will show that for any l there is a solution $\tilde{u}(\,\cdot\,, l) \in L^2_{-\delta}(\mathbb{R})$ of the ODE

$$(-\partial_1^2 + 2\tau\partial_1 - \tau^2 + \lambda_l)\tilde{u}(\,\cdot, l) = \tilde{f}(\,\cdot\,, l) \tag{4.10}$$

satisfying the norm estimate

$$\|\tilde{u}(\,\cdot\,, l)\|_{L^2_{-\delta}(\mathbb{R})} \leq \frac{C_0}{\tau + \sqrt{\lambda_l}}\|\tilde{f}(\,\cdot\,, l)\|_{L^2_\delta(\mathbb{R})}. \tag{4.11}$$

In fact, using the factorization to first order equations given after (4.4), the ODE for $\tilde{u}(\,\cdot\,,l)$ becomes

$$(\partial_1 - \tau - \sqrt{\lambda_l})(\partial_1 - \tau + \sqrt{\lambda_l})\tilde{u}(\,\cdot\,,l) = -\tilde{f}(\,\cdot\,,l).$$

Since $\tilde{f}(\,\cdot\,,l) \in L^2_\delta(\mathbb{R})$, Proposition 4.4 shows there is a unique solution given by

$$\tilde{u}(\,\cdot\,,l) := -S_{\tau-\sqrt{\lambda_l}} S_{\tau+\sqrt{\lambda_l}} \tilde{f}(\,\cdot\,,l). \tag{4.12}$$

Since $\tau - \sqrt{\lambda_l} \neq 0$ and $\tau + \sqrt{\lambda_l} \geq 1$ by the assumptions on τ, the estimates in Proposition 4.4 yield (4.11).

Step 3. With $\tilde{u}(\,\cdot\,,l)$ as above, define

$$u_N(x_1, x') := \sum_{l=1}^{N} \tilde{u}(x_1, l)\phi_l(x').$$

Our objective is to show that as $N \to \infty$, u_N converges in $L^2_{-\delta}(T)$ to a function u which is a weak solution of the equation (4.9) and satisfies

$$\|u\|_{L^2_{-\delta}(T)} \leq \frac{C_0}{\tau} \|f\|_{L^2_\delta(T)}.$$

If $N' > N$, the orthogonality argument in Step 1 and the estimate (4.11) show that

$$\|u_{N'} - u_N\|^2_{L^2_{-\delta}(T)} = \sum_{l=N}^{N'-1} \|\tilde{u}(\,\cdot\,,l)\|^2_{L^2_{-\delta}(\mathbb{R})} \leq \left(\frac{C_0}{\tau}\right)^2 \sum_{l=N}^{N'-1} \|\tilde{f}(\,\cdot\,,l)\|^2_{L^2_\delta(\mathbb{R})}.$$

Since $f \in L^2_\delta(T)$ the last expression converges to zero as $N, N' \to \infty$. This shows that (u_N) is a Cauchy sequence in $L^2_{-\delta}(T)$, hence converges to a function $u \in L^2_{-\delta}(T)$.

Using that $-\Delta_{g_0}\phi_l = \lambda_l \phi_l$, we have by (4.10)

$$\begin{aligned}(-\partial_1^2 + 2\tau\partial_1 - \tau^2 - \Delta_{g_0})u_N &= \sum_{l=1}^{N} (-\partial_1^2 + 2\tau\partial_1 - \tau^2 + \lambda_l)\tilde{u}(x_1, l)\phi_l(x') \\ &= \sum_{l=1}^{N} \tilde{f}(x_1, l)\phi_l(x').\end{aligned}$$

The right hand side converges to f in $L^2_\delta(T)$ as $N \to \infty$. Integrating against a test function in $C_c^\infty(T^{\mathrm{int}})$, we see that u is indeed a weak solution of (4.9). The norm estimate follows from orthogonality and (4.11):

$$\begin{aligned}\|u\|^2_{L^2_{-\delta}(T)} &= \sum_{l=1}^{\infty}\|\tilde{u}(\,\cdot\,,l)\|^2_{L^2_{-\delta}(\mathbb{R})} \le \left(\frac{C_0}{\tau}\right)^2 \sum_{l=1}^{\infty}\|\tilde{f}(\,\cdot\,,l)\|^2_{L^2_{\delta}(\mathbb{R})} \\ &\le \left(\frac{C_0}{\tau}\right)^2 \|f\|^2_{L^2_{\delta}(T)}.\end{aligned}$$

Step 4. It remains to show that $u \in H^1_{-\delta,0}(T)$ and

$$\|u\|_{H^1_{-\delta}(T)} \le C_0\|f\|_{L^2_\delta(T)}.$$

This can be done by looking at the first order derivatives in x_1 and x' separately. By the definition (4.12) of $\tilde{u}(\,\cdot\,,l)$ (where of course $S_{\tau-\sqrt{\lambda_l}}$ and $S_{\tau+\sqrt{\lambda_l}}$ can be interchanged) and the definition of S_a, we have

$$\partial_1\tilde{u}(\,\cdot\,,l) = (\tau+\sqrt{\lambda_l})\tilde{u}(\,\cdot\,,l) - S_{\tau-\sqrt{\lambda_l}}\tilde{f}(\,\cdot\,,l).$$

Then (4.11) and Proposition 4.4 imply

$$\|\partial_1\tilde{u}(\,\cdot\,,l)\|_{L^2_{-\delta}(\mathbb{R})} \le C_0\|\tilde{f}(\,\cdot\,,l)\|_{L^2_\delta(\mathbb{R})}.$$

Orthogonality shows that $\|\partial_1 u\|_{L^2_{-\delta}(T)} \le C_0\|f\|_{L^2_\delta(T)}$.

For the x' derivatives we use the exterior derivative $d_{x'}$ in (M_0,g_0). Since u_N vanishes on $\mathbb{R}\times\partial M_0$, we have

$$\begin{aligned}\|d_{x'}u_N\|^2_{L^2_{-\delta}(T)} &= \int_{-\infty}^{\infty}\langle x_1\rangle^{-2\delta}\langle d_{x'}u_N, d_{x'}\bar{u}_N\rangle_{M_0}\,dx_1 \\ &= \int_{-\infty}^{\infty}\langle x_1\rangle^{-2\delta}\langle(-\Delta_{g_0}u_N),\bar{u}_N\rangle_{M_0}\,dx_1 \\ &= \int_{-\infty}^{\infty}\sum_{l=1}^{N}\langle x_1\rangle^{-2\delta}\lambda_l|\tilde{u}(\,\cdot\,,l)|^2\,dx_1 \\ &= \sum_{l=1}^{N}\lambda_l\|\tilde{u}(\,\cdot\,,l)\|^2_{L^2_{-\delta}(\mathbb{R})}.\end{aligned}$$

Orthogonality and (4.11) give the estimate $\|d_{x'}u_N\|^2_{L^2_{-\delta}(T)} \le C_0\|f\|^2_{L^2_\delta(T)}$. An argument using Cauchy sequences shows that $d_{x'}u_N$ converges in $L^2_{-\delta}(T)$, hence also $d_{x'}u \in L^2_{-\delta}(T)$ and $\|d_{x'}u\|_{L^2_{-\delta}(T)} \le C_0\|f\|_{L^2_\delta(T)}$.

We have proved that $u \in H^1_{-\delta}(T)$ with the right norm estimate. It is now enough to note that $u_N \in H^1_{-\delta,0}(T)$, and the same is true for the limit u since this space is closed in $H^1_{-\delta}(T)$. $\square$

We have now completed the proof of Theorem 4.1 in the case where $c = 1$ and $V = 0$. In fact, the combination of Propositions 4.3 and 4.6 immediately shows the existence of a solution operator G_τ for the conjugated Laplace-Beltrami equation with metric $g = e \oplus g_0$.

PROPOSITION 4.7. *(Solution operator for $V = 0$) Let $\delta > 1/2$. If $|\tau| \geq 1$ and $\tau^2 \notin Spec(-\Delta_{g_0})$, there is a bounded operator*

$$G_\tau : L^2_\delta(T) \to H^1_{-\delta,0}(T)$$

such that $u = G_\tau f$ is the unique solution in $H^1_{-\delta,0}(T)$ of the equation

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = f \quad \textit{in } T.$$

This operator satisfies

$$\|G_\tau f\|_{L^2_{-\delta}(T)} \leq \frac{C_0}{|\tau|}\|f\|_{L^2_\delta(T)},$$
$$\|G_\tau f\|_{H^1_{-\delta}(T)} \leq C_0\|f\|_{L^2_\delta(T)}.$$

It is now an easy matter to prove Theorem 4.1 also with a nonzero potential V by using a perturbation argument.

PROOF OF THEOREM 4.1. We assume, as we may, that $c \equiv 1$. Let us first consider uniqueness. Assume that $u \in H^1_{-\delta,0}(T)$ satisfies

$$e^{\tau x_1}(-\Delta_g + V)e^{-\tau x_1}u = 0 \quad \text{in } T.$$

This can be written as

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = -Vu \quad \text{in } T.$$

By the assumption $\langle x_1\rangle^{2\delta}V \in L^\infty(T)$, the right hand side is in $L^2_\delta(T)$. The uniqueness part of Proposition 4.7 implies

$$u = -G_\tau(Vu).$$

The norm estimates for G_τ give

$$\|u\|_{L^2_{-\delta}(T)} \leq \frac{C_0\|\langle x_1\rangle^{2\delta}V\|_{L^\infty(T)}}{|\tau|}\|u\|_{L^2_{-\delta}(T)}.$$

Thus, if we choose

$$\tau_0 := \max(2C_0\|\langle x_1\rangle^{2\delta}V\|_{L^\infty(T)}, 1), \tag{4.13}$$

then the condition $|\tau| \geq \tau_0$ will imply $\|u\|_{L^2_{-\delta}(T)} \leq \frac{1}{2}\|u\|_{L^2_{-\delta}(T)}$, showing that $u \equiv 0$.

As for existence, we seek a solution of the equation

$$e^{\tau x_1}(-\Delta_g + V)e^{-\tau x_1}u = f \quad \text{in } T$$

in the form $u = G_\tau \tilde{f}$ for some $\tilde{f} \in L^2_\delta(T)$. Inserting this expression in the equation and using that G_τ is the inverse of the conjugated Laplace-Beltrami operator, we see that $\tilde{f}$ should satisfy

$$(\mathrm{Id} + VG_\tau)\tilde{f} = f \quad \text{in } T.$$

Now if $|\tau| \geq \tau_0$ with τ_0 as in (4.13), we have

$$\|VG_\tau\|_{L^2_\delta(T)\to L^2_\delta(T)} \leq \frac{C_0\|\langle x_1\rangle^{2\delta}V\|_{L^\infty(T)}}{|\tau|} \leq \frac{1}{2}.$$

Thus $\mathrm{Id} + VG_\tau$ is invertible on $L^2_\delta(T)$, with norm of the inverse ≤ 2. It follows that $u := G_\tau(\mathrm{Id} + VG_\tau)^{-1}f$ is a solution with the required properties. $\square$

Exercise 4.3. Prove that the solution construction in Theorem 4.1 is in fact in $H^2_{-\delta}(T)$ and satisfies $\|u\|_{H^2_{-\delta}(T)} \leq C_0|\tau|\|f\|_{L^2_\delta(T)}$.

Exercise 4.4. Prove Theorem 4.1 in the more general case where the Schrödinger operator $-\Delta_g + V$ with $\langle x_1\rangle^{2\delta}V \in L^\infty(T)$ is replaced by a Helmholtz operator $-\Delta_g + V - k^2$ where $k > 0$ is fixed.

CHAPTER 5

Uniqueness result

In this chapter we will prove a uniqueness result for the inverse problem considered in the introduction. The result will be proved for the case of the Schrödinger equation in a compact manifold (M, g). The method, as discussed in Chapter 3, is to show that the set of products $\{u_1 u_2\}$ of solutions to two Schrödinger equations is dense in $L^1(M)$. The special solutions which will be used to prove the density statement have the form

$$u = e^{\pm\tau\varphi}(m + r_0).$$

The starting point for constructing such solutions is an LCW φ. For this reason we will need to work in manifolds which admit LCWs. Thus we will assume that (M, g) is contained in a cylinder (T, g) where $T = \mathbb{R} \times M_0$ and $g = c(e \oplus g_0)$, which is roughly equivalent to M having an LCW by the results in Chapter 3.

However, the existence of an LCW is only a starting point for the solution of the inverse problem. One also needs construct the term m so that $e^{\pm\tau\varphi}m$ is an approximate solution, which can be corrected into an exact solution by the term r_0 obtained from solving an inhomogeneous equation as in Chapter 4. Finally, one needs to do this construction so that the density of the products $\{u_1 u_2\}$ can be proved by using the special solutions. In Euclidean space one typically employs the Fourier transform, which is not immediately available in (M, g).

We will use a hybrid method which involves the Fourier transform in the x_1 variable where it is available, and integrals over geodesics in the x' variables. In fact, we will choose the functions m to concentrate near fixed geodesics in (M_0, g_0). The uniqueness theorem will then rely on the result that a function in M_0 can be determined from its integrals over geodesics. At present, such a result is only known under strong restrictions on the geodesic flow of (M_0, g_0). One such restriction is that (M_0, g_0) is *simple*, meaning roughly that any two points can be connected by a unique length minimizing geodesic.

Leaving the precise definition of simple manifolds to Section 5.2, we now define the class of admissible manifolds for which we can prove uniqueness results for inverse problems. There are three conditions: the first one requiring the dimension to be at least three (the case of 2D manifolds requires quite different methods), the second stating that the manifold should admit an LCW, and the third stating that the transversal manifold (M_0, g_0) satisfies a restriction ensuring that functions are determined by their integrals over geodesics.

Definition. A compact manifold (M, g) with smooth boundary is called *admissible* if

(a) $\dim(M) \geq 3$,
(b) $(M, g) \subset\subset (T, g)$ where $T = \mathbb{R} \times M_0$ and $g = c(e \oplus g_0)$ with $c > 0$ a smooth positive function and e the Euclidean metric on $\mathbb{R}$, and
(c) (M_0, g_0) is a simple $(n-1)$-dimensional manifold.

The main uniqueness result is as follows. Recall that we implicitly assume that all DN maps are well defined.

Theorem 5.1. *(Global uniqueness) Let (M, g) be an admissible manifold, and assume that V_1 and V_2 are continuous functions on M. If $\Lambda_{g,V_1} = \Lambda_{g,V_2}$, then $V_1 = V_2$.*

In fact, it is enough to prove the theorem for admissible manifolds where the conformal factor is constant and V_1 and V_2 are in $C_c(M^{\mathrm{int}})$. In the proofs below, we will work under these assumptions. We now give a sketch how to make this reduction.

Suppose (M, g) is admissible and $g = c\tilde{g}$ with $\tilde{g} = e \oplus g_0$, and assume that $\Lambda_{g,V_1} = \Lambda_{g,V_2}$. Note that we are free to assume that $c = 1$ outside a small neighborhood of M in T. A boundary determination result [**4**, Theorem 8.4] shows that $V_1|_{\partial M} = V_2|_{\partial M}$. Extending V_1, V_2 to a slightly larger admissible manifold $(\tilde{M}, g)$ so that $c = 1$ and $V_1 = V_2 = 0$ near $\partial\tilde{M}$, it is not hard to see that $\Lambda_{g,V_1} = \Lambda_{g,V_2}$ for the DN maps in $(\tilde{M}, g)$. Now by the conformal scaling law for Δ_g, it holds that

$$\Lambda_{c\tilde{g},V_j} = \Lambda_{\tilde{g},c(V_j - q_c)}$$

where $q_c = c^{\frac{n-2}{4}} \Delta_{c\tilde{g}}(c^{-\frac{n-2}{4}})$. Thus $\Lambda_{\tilde{g},V_1} = \Lambda_{\tilde{g},V_2}$ for the DN maps in $(\tilde{M}, \tilde{g})$, which completes the reduction.

5.1. Complex geometrical optics solutions

Here we will construct the special solutions, also called *complex geometrical optics solutions*, to the Schrödinger equation. The first step is to construct approximate solutions

$$u_0 = e^{-\tau\Phi}a$$

where $\tau > 0$ is a large parameter, $\Phi \in C^\infty(M)$ is a complex function (the complex phase), and a is smooth complex function on M (the complex amplitude). Note that we have replaced the real function φ with a complex function Φ. In fact the real part of Φ is later taken to be an LCW φ.

We extend the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ as a $\mathbb{C}$-bilinear form to complex tangent and cotangent vectors. This means that for $\xi, \eta, \xi', \eta' \in T_p^*M$,

$$\langle \xi + i\eta, \xi' + i\eta'\rangle := \langle \xi, \xi'\rangle - \langle \eta, \eta'\rangle + i(\langle \xi, \eta'\rangle + \langle \eta, \xi'\rangle).$$

Note that $\langle\,\cdot\,,\,\cdot\,\rangle$ is not a Hermitian inner product, since there are nonzero complex vectors whose inner product with itself is zero.

With this notation, we have the following analog of the computation in Lemma 3.2 (just replace φ by Φ).

LEMMA 5.2. *(Expression for conjugated Schrödinger operator)*

$$\begin{aligned} e^{\tau\Phi}(-\Delta_g + V)e^{-\tau\Phi}v = -\tau^2\langle d\Phi, d\Phi\rangle v \\ + \tau\left[2\langle d\Phi, dv\rangle + (\Delta_g\Phi)v\right] + (-\Delta_g + V)v. \end{aligned}$$

Note that this result gives an expansion of the conjugated operator $e^{\tau\Phi}(-\Delta_g+V)e^{-\tau\Phi}$ in terms of powers of τ. We will look for approximate solutions $u_0 = e^{-\tau\Phi}a$ such that the terms with highest powers of τ go away. This leads to equations for Φ and a, and also an equation for the correction term r_0 when one looks for the exact solution u corresponding to u_0. The next result follows from Lemma 5.2.

PROPOSITION 5.3. *(Equations) Let (M, g) be a compact manifold with boundary and let $V \in L^\infty(M)$. The function $u = e^{-\tau\Phi}(a + r_0)$ is a solution of $(-\Delta_g + V)u = 0$ in M, provided that in M*

$$\langle d\Phi, d\Phi\rangle = 0, \tag{5.1}$$

$$2\langle d\Phi, da\rangle + (\Delta_g\Phi)a = 0, \tag{5.2}$$

$$e^{\tau\Phi}(-\Delta_g + V)e^{-\tau\Phi}r_0 = (\Delta_g - V)a. \tag{5.3}$$

The last result is analogous the (real) geometrical optics method, or the WKB method, for constructing solutions to various equations. The main difference to the standard setting is that we need to consider complex quantities. Here (5.1) is called a *complex eikonal equation*, that is, a certain nonlinear first order equation for the complex phase Φ. The equation (5.2) is a *complex transport equation*, which is a linear first order equation for the amplitude a. The last equation (5.3) is an inhomogeneous equation for the correction term r_0.

Writing $\Phi = \varphi + i\psi$ where φ and ψ are real, the equation (5.3) becomes

$$e^{\tau\varphi}(-\Delta_g + V)e^{-\tau\varphi}(e^{-i\tau\psi}r_0) = e^{-i\tau\psi}(\Delta_g - V)a.$$

This equation can be solved by Theorem 4.1 if φ is an LCW and the manifold has an underlying product structure. Using that ψ is real we have $\|e^{-i\tau\psi}v\|_{L^2(M)} = \|v\|_{L^2(M)}$, so the terms $e^{-i\tau\psi}$ will not change the resulting L^2 estimates.

We now assume that (M, g) is admissible, and further that $c \equiv 1$ which is possible by the reduction above. Thus (M, g) is embedded in the cylinder (T, g) where $T = \mathbb{R} \times M_0$ and $g = e \oplus g_0$, and further $(M_0, g_0) \subset\subset (U, g_0)$ with $(\overline{U}, g_0)$ simple. In the coordinates $x = (x_1, x')$,

$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}.$$

We also assume that $\mathrm{Re}(\Phi) = \varphi$ where $\varphi(x_1, x') := x_1$ is the natural LCW in the cylinder.

Eikonal equation. Writing $\Phi = \varphi + i\psi$ where φ and ψ are real valued, the complex eikonal equation (5.1) becomes the pair of equations

$$|d\psi|^2 = |d\varphi|^2, \quad \langle d\varphi, d\psi\rangle = 0. \tag{5.4}$$

Using that $\varphi(x) = x_1$ and the special form of the metric, these equations become

$$|d\psi|^2 = 1, \quad \partial_1\psi = 0.$$

The second equation just means that ψ should be independent of x_1, that is, $\psi = \psi(x')$. Thus we have reduced matters to solving a (real) eikonal equation in M_0:

$$|d\psi|_{g_0}^2 = 1 \quad \text{in } M_0.$$

Such an equation does not have global smooth solutions on a general manifold (M_0, g_0). However, in our case where (M_0, g_0) is assumed to be simple (see Section 5.2), there are many global smooth solutions. It is enough to choose some point $\omega \in U \setminus M_0$ and to take

$$\psi(x_1, r, \theta) = \psi_\omega(x_1, r, \theta) := r$$

where (r, θ) are polar normal coordinates in (U, g_0) with center ω. Since $|dr|_{g_0} = 1$ on the maximal domain where polar normal coordinates are defined (excluding the center), this gives a smooth solution in M.

In fact, if $x = (x_1, r, \theta)$ are coordinates in T where (r, θ) are polar normal coordinates in (U, g_0) with center ω, then the form of the metric g_0 in polar normal coordinates shows that

$$g(x_1, r, \theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g_1(r, \theta) \end{pmatrix} \tag{5.5}$$

for some $(n-2) \times (n-2)$ positive definite matrix g_1. This gives the coordinate representation

$$\Phi(x_1, r, \theta) = \Phi_\omega(x_1, r, \theta) := x_1 + ir.$$

Remark 5.4. For $n = 2$ the complex eikonal equation, which is equivalent to the pair (5.4), just says that φ and ψ should be (anti)-conjugate harmonic functions, so that Φ should be (anti)holomorphic. In dimensions $n \geq 3$ solutions of the complex eikonal equation can be considered as analogs in a certain sense of (anti)holomorphic functions. In our setting, using the given coordinates, Φ is just $x_1 + ir$ which can be considered as a complex variable z and hence also as a holomorphic function.

Transport equation. Having obtained the complex phase $\Phi = \varphi + i\psi = x_1 + ir$, it is not difficult to solve the complex transport equation. Using the coordinates (x_1, r, θ) and the special form (5.5) for the metric, we see that

$$\langle d\Phi, da \rangle = g^{jk} \partial_j \Phi \partial_k a = (\partial_1 + i\partial_r) a$$

and

$$\begin{aligned}\Delta_g \Phi &= |g|^{-1/2}\partial_j(|g|^{1/2} g^{jk}\partial_k(x_1 + ir)) \\ &= |g|^{-1/2}\partial_r(|g|^{1/2} i) \\ &= \frac{1}{2}(\partial_1 + i\partial_r)(\log|g|).\end{aligned}$$

The transport equation (5.2) now has the form

$$(\partial_1 + i\partial_r)a + (\partial_1 + i\partial_r)(\log|g|^{1/4})a = 0.$$

Multiplying by the integrating factor $|g|^{1/4}$, we obtain the equivalent equation

$$(\partial + i\partial_r)(|g|^{1/4}a) = 0.$$

Thus the complex amplitudes satisfying (5.2) have the form

$$a(x_1, r, \theta) = |g|^{-1/4} a_0(x_1, r, \theta)$$

where a_0 is a smooth function in M satisfying $(\partial_1 + i\partial_r)a_0 = 0$.

Inhomogeneous equation. Given Φ and a, the final equation (5.3) in the present setting becomes

$$e^{\tau x_1}(-\Delta_g + V)e^{-\tau x_1}(e^{-i\tau r} r_0) = f \quad \text{in } M$$

where $f := e^{-i\tau r}(\Delta_g - V)a$. We extend V and f by zero to T, and consider the equation

$$e^{\tau x_1}(-\Delta_g + V)e^{-\tau x_1} v = f \quad \text{in } T.$$

If $|\tau|$ is large and $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$, this equation has a unique solution $v \in H^1_{-\delta,0}(T)$ by Theorem 4.1. It satisfies for any $\delta > 1/2$

$$\|v\|_{L^2_{-\delta}(T)} \leq \frac{C_0}{|\tau|}\|f\|_{L^2_\delta(T)}.$$

Define $r_0 := e^{i\tau r} v|_M$. Then $r_0 \in H^1(M)$ and

$$\|r_0\|_{L^2(M)} \leq \frac{C_0}{|\tau|}\|a\|_{H^2(M)}.$$

Also, r_0 satisfies (5.3) by construction.

We collect the results of the preceding arguments in the next proposition.

PROPOSITION 5.5. *(Complex geometrical optics solutions) Assume (M, g) is an admissible manifold embedded in (T, g), where $T = \mathbb{R} \times M_0$ and $g = e \oplus g_0$ and where $(M_0, g_0) \subset\subset (\overline{U}, g_0)$ are simple manifolds. Let also $V \in L^\infty(M)$. There are $C_0, \tau_0 > 0$ such that whenever*

$$|\tau| \geq \tau_0 \quad \textit{and} \quad \tau^2 \notin \mathrm{Spec}(-\Delta_{g_0}),$$

then for any $\omega \in U \smallsetminus M_0$ and for any smooth function a_0 in M with $(\partial_1 + i\partial_r)a_0 = 0$, where (x_1, r, θ) are coordinates in M such that (r, θ) are polar normal coordinates in (U, g_0) with center ω, there is a solution

$$u = e^{-\tau(x_1+ir)}(|g|^{-1/4}a_0 + r_0)$$

of the equation $(-\Delta_g + V)u = 0$ in M, such that

$$\|r_0\|_{L^2(M)} \leq \frac{C_0}{|\tau|}\|a_0\|_{H^2(M)}.$$

We can now complete the proof of Theorem 5.1, modulo the following statement on the attenuated geodesic ray transform which will be discussed in the next section.

THEOREM. *(Injectivity for the attenuated geodesic ray transform) Let (M_0, g_0) be a simple manifold. There exists $\varepsilon > 0$ such that for any $\lambda \in (-\varepsilon, \varepsilon)$, if a function $f \in C(M_0)$ satisfies*

$$\int_\gamma e^{-\lambda t} f(\gamma(t))\, dt$$

for any maximal geodesic γ going from ∂M_0 into M_0, then $f \equiv 0$.

PROOF OF THEOREM 5.1. We make the reduction described after Theorem 5.1 to the case where $c \equiv 1$ and $V_1, V_2 \in C_c(M^{\mathrm{int}})$. The assumption that $\Lambda_{g,V_1} = \Lambda_{g,V_2}$ implies that

$$\int_M (V_1 - V_2)u_1 u_2\, dV = 0$$

for any $u_j \in H^1(M)$ with $(-\Delta_g + V_j)u_j = 0$ in M.

We use Proposition 5.5 and choose u_j to be solutions of the following form. Let ω be a fixed point in $U \smallsetminus M_0$, let (x_1, r, θ) be coordinates near M such (r, θ) are polar normal coordinates in (U, g_0) with center ω, and let λ be a fixed real number and $b = b(\theta) \in C^\infty(S^{n-2})$ a fixed function. Then, for $\tau > 0$ large enough and outside a discrete set, we can choose u_j of the form

$$u_1 = e^{-\tau(x_1+ir)}(|g|^{-1/4}e^{i\lambda(x_1+ir)}b(\theta) + r_1),$$
$$u_2 = e^{\tau(x_1+ir)}(|g|^{-1/4} + r_2).$$

Note that the functions $e^{i\lambda(x_1+ir)}b(\theta)$ and 1 are holomorphic in the (x_1, r) variables, so we indeed have solutions of this form. Further, $\|r_j\|_{L^2(M)} \leq C/\tau$.

Inserting the solutions in the integral identity and letting $\tau \to \infty$ outside a discrete set, we obtain

$$\int_M (V_1 - V_2)|g|^{-1/2}e^{i\lambda(x_1+ir)}b(\theta)\, dV_g = 0.$$

Since V_1 and V_2 are compactly supported, the integral can be taken over the cylinder T. Using the (x_1, r, θ) coordinates in T and the fact that $dV_g = |g(x_1, r, \theta)|^{1/2}\, dx_1\, dr\, d\theta$, this implies that

$$\int_{S^{n-2}} \left[\int_{-\infty}^{\infty}\int_0^{\infty}(V_1 - V_2)(x_1, r, \theta)e^{i\lambda(x_1+ir)}\, dx_1\, dr\right] b(\theta)\, d\theta = 0.$$

The last statement is valid for any fixed $b \in C^\infty(S^{n-2})$. We can choose b to resemble a delta function at a fixed direction θ_0 in S^{n-2}, and varying b will then imply that the quantity in brackets vanishes for all θ_0. This is the point where we have chosen the solution u_1 to approximately concentrate near a fixed geodesic, corresponding to a fixed direction in S^{n-2}, in the transversal manifold (M_0, g_0).

We have proved that

$$\int_0^{\infty} e^{-\lambda r}\left[\int_{-\infty}^{\infty}(V_1 - V_2)(x_1, r, \theta)e^{i\lambda x_1}\, dx_1\right] dr = 0, \quad \text{for all } \theta.$$

Denote the quantity in brackets by $f_\lambda(r, \theta)$. Then f_λ is a smooth function in (M_0, g_0) compactly supported in M_0^{int}, and the curve $\gamma_{\omega,\theta} : r \mapsto (r, \theta)$ is a geodesic in (U, g_0) issued from the point ω in direction θ. This shows that

$$\int_0^{\infty} e^{-\lambda r} f_\lambda(\gamma_{\omega,\theta}(r))\, dr = 0$$

for all $\omega \in U \smallsetminus M_0$ and for all directions θ. Letting ω approach the boundary of M_0 and varying θ, the last result implies that

$$\int_\gamma e^{-\lambda t} f_\lambda(\gamma(t))\, dt = 0$$

for all geodesics γ starting from points of ∂M_0 which are maximal in the sense that γ is defined for the maximal time until it exits M_0.

The injectivity result for the attenuated geodesic ray transform, stated just before this proof, shows that there is $\varepsilon > 0$ such that for any $\lambda \in (-\varepsilon, \varepsilon)$, the function f_λ is identically zero on M_0. Thus for $|\lambda| < \varepsilon$,

$$\int_{-\infty}^{\infty} (V_1 - V_2)(x_1, r, \theta) e^{i\lambda x_1} \, dx_1 = 0, \quad \text{for any fixed } r, \theta.$$

If (r, θ) is fixed then the function $x_1 \mapsto (V_1 - V_2)(x_1, r, \theta)$ is compactly supported on the real line, and the last result says that its Fourier transform vanishes for $|\lambda| < \varepsilon$. But by the Paley-Wiener theorem the Fourier transform is analytic, which is only possible if $(V_1 - V_2)(\,\cdot\,, r, \theta) = 0$ on the real line. This is true for any fixed (r, θ), showing that $V_1 = V_2$ as required. $\square$

5.2. Geodesic ray transform

In this section we will give some arguments related to the injectivity result for the attenuated geodesic ray transform, which was used in the proof of the global uniqueness theorem. The treatment will be very sketchy and not self-contained, but hopefully it will give an idea about why such a result would be true.

Explicit inversion methods. To set the stage and to obtain some intuition to the problem, we first consider the classical question of inverting the Radon transform in $\mathbb{R}^2$. This is the transform which integrates a function $f \in C_c^\infty(\mathbb{R}^2)$ over all lines, and can be expressed as follows:

$$Rf(s, \omega) := \int_{-\infty}^{\infty} f(s\omega^\perp + t\omega) \, dt, \quad s \in \mathbb{R}, \omega \in S^1.$$

Here $\omega^\perp$ is the vector in S^1 obtained by rotating ω counterclockwise by $90°$.

There is a well-known relation between Rf and the Fourier transform $\hat{f}$. We denote by $\widehat{Rf}(\,\cdot\,, \omega)$ the Fourier transform of Rf with respect to s.

Proposition 5.6. *(Fourier slice theorem)*

$$\widehat{Rf}(\sigma, \omega) = \hat{f}(\sigma\omega^\perp).$$

Proof. Parametrizing $\mathbb{R}^2$ by $y = s\omega^\perp + t\omega$, we have

$$\widehat{Rf}(\sigma,\omega) = \int_{-\infty}^{\infty} e^{-i\sigma s} \int_{-\infty}^{\infty} f(s\omega^\perp + t\omega)\, dt\, ds = \int_{\mathbb{R}^2} e^{-i\sigma y\cdot\omega^\perp} f(y)\, dy$$
$$= \hat{f}(\sigma\omega^\perp).$$

□

This result gives the first proof of injectivity of Radon transform: if $f \in C_c^\infty(\mathbb{R}^2)$ is such that $Rf \equiv 0$, then $\hat{f} \equiv 0$ and consequently $f \equiv 0$. To obtain a different inversion formula, and for later purposes, we will consider the adjoint of R. This is obtained by computing for $f \in C_c^\infty(\mathbb{R}^2)$ and $h \in C^\infty(\mathbb{R}\times S^1)$ that

$$\begin{aligned}(Rf,h)_{\mathbb{R}\times S^1} &= \int_{-\infty}^{\infty}\int_{S^1} Rf(s,\omega)h(s,\omega)\, d\omega\, ds \\ &= \int_{-\infty}^{\infty}\int_{S^1}\int_{-\infty}^{\infty} f(s\omega^\perp + t\omega)h(s,\omega)\, dt\, d\omega\, ds \\ &= \int_{\mathbb{R}^2} f(y)\left(\int_{S^1} h(y\cdot\omega^\perp,\omega)\, d\omega\right) dy.\end{aligned}$$

Thus the adjoint of R is the operator

$$R^* : C^\infty(\mathbb{R}\times S^1) \to C^\infty(\mathbb{R}^2), \quad R^*h(y) = \int_{S^1} h(y\cdot\omega^\perp,\omega)\, d\omega.$$

Proposition 5.7. *(Fourier transform of R^*) Letting $\hat{\xi} = \frac{\xi}{|\xi|}$,*

$$(R^*h)\hat{\ }(\xi) = \frac{2\pi}{|\xi|}\left(\hat{h}(|\xi|,-\hat{\xi}^\perp) + \hat{h}(-|\xi|,\hat{\xi}^\perp)\right).$$

Proof. We will make a formal computation (which is not difficult to justify). Using again the parametrization $y = s\omega^\perp + t\omega$,

$$\begin{aligned}(R^*h)\hat{\ }(\xi) &= \int_{\mathbb{R}^2}\int_{S^1} e^{-iy\cdot\xi} h(y\cdot\omega^\perp,\omega)\, d\omega\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{S^1} e^{-is\omega^\perp\cdot\xi} e^{-it\omega\cdot\xi} h(s,\omega)\, d\omega\, ds\, dt \\ &= \int_{S^1} \hat{h}(\omega^\perp\cdot\xi,\omega)\left(\int_{-\infty}^{\infty} e^{-it\omega\cdot\xi}\, dt\right) d\omega.\end{aligned}$$

The quantity in the parentheses is just $\frac{2\pi}{|\xi|}\delta_0(\omega\cdot\hat{\xi})$ where δ_0 is the Dirac delta function at the origin. Since $\omega\cdot\hat{\xi}$ is zero exactly when $\omega = \pm\hat{\xi}^\perp$, the result follows. □

The Radon transform in $\mathbb{R}^2$ satisfies the symmetry $Rf(-s,-\omega) = Rf(s,\omega)$, and the Fourier slice theorem implies

$$(R^*Rf)\hat{}(\xi) = \frac{4\pi}{|\xi|}\widehat{Rf}(|\xi|, -\hat{\xi}^\perp) = \frac{4\pi}{|\xi|}\hat{f}(\xi).$$

This shows that the normal operator R^*R is a classical pseudodifferential operator of order -1 in $\mathbb{R}^2$, and also gives an inversion formula.

PROPOSITION 5.8. *(Normal operator) One has*

$$R^*R = 4\pi(-\Delta)^{-1/2},$$

and f can be recovered from Rf by the formula

$$f = \frac{1}{4\pi}(-\Delta)^{1/2}R^*Rf.$$

The last result is an example of an explicit inversion method for the Radon transform in the Euclidean plane, based on the Fourier transform. Similar methods are available for the Radon transform on manifolds with many symmetries where variants of the Fourier transform exist (see [**8**] and other books of Helgason for results of this type). However, for manifolds which do not have symmetries, such as small perturbations of the Euclidean metric, explicit transforms are usually not available and other inversion methods are required.

Pseudodifferential methods. Let (M,g) be a compact manifold with smooth boundary, assumed to be embedded in a compact manifold (N,g) without boundary. We parametrize geodesics by points in the *unit sphere bundle*, defined by

$$SM := \bigvee_{x\in M} S_xM, \quad S_xM := \{\xi \in T_xM \,;\, |\xi| = 1\}.$$

If $(x,\xi) \in SM$ we denote by $\gamma(t,x,\xi)$ the geodesic in N which starts at the point x in direction ξ, that is,

$$D_{\dot\gamma}\dot\gamma = 0, \quad \gamma(0,x,\xi) = x, \quad \dot\gamma(0,x,\xi) = \xi.$$

Let $\tau(x,\xi)$ be the first time when $\gamma(t,x,\xi)$ exits M,

$$\tau(x,\xi) := \inf\{t > 0\,;\, \gamma(t,x,\xi) \in N \smallsetminus M\}.$$

We assume that (M,g) is *nontrapping*, meaning that $\tau(x,\xi)$ is finite for any $(x,\xi) \in SM$.

The *geodesic ray transform* of a function $f \in C^\infty(M)$ is defined by

$$If(x,\xi) := \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,dt, \quad (x,\xi) \in \partial(SM).$$

Thus, If gives the integral of f over any maximal geodesic in M starting from ∂M, such geodesics being parametrized by points of $\partial(SM) = \{(x,\xi) \in SM\,;\, x \in \partial M\}$.

So far, we have not imposed any restrictions on the behavior of geodesics in (M,g) other than the nontrapping condition. However, injectivity and inversion results for If are only known under strong geometric restrictions. One class of manifolds where such results have been proved is the following. From now on the treatment will be sketchy, and we refer to [**4**], [**2**], [**18**] for more details.

Definition. A compact manifold (M,g) with boundary is called *simple* if

(a) for any point $p \in M$, the exponential map $\exp_p$ is a diffeomorphism from its maximal domain in T_pM onto M, and
(b) the boundary ∂M is strictly convex.

Several remarks are in order. A diffeomorphism is, as earlier, a homeomorphism which together with its inverse is smooth up to the boundary. The maximal domain of $\exp_p$ is starshaped, and the fact that $\exp_p$ is a diffeomorphism onto M thus implies that M is diffeomorphic to a closed ball. The last fact uses that τ is smooth in $S(M^{\mathrm{int}})$. This is a consequence of strict convexity, which is precisely defined as follows:

Definition. Let (M,g) be a compact manifold with boundary. We say that ∂M is *stricly convex* if the second fundamental form $l_{\partial M}$ is positive definite. Here $l_{\partial M}$ is the 2-tensor on ∂M defined by

$$l_{\partial M}(X,Y) = -\langle D_X \nu, Y\rangle, \qquad X,Y \in C^\infty(\partial M, T(\partial M)),$$

where ν is the outer unit normal to ∂M.

Alternatively, the boundary is strictly convex iff any geodesic in N starting from a point $x \in \partial M$ in a direction tangent to ∂M stays outside M for small positive and negative times. This implies that any maximal geodesic going from ∂M into M stays inside M except for its endpoints, which corresponds to the usual notion of strict convexity.

If (M, g) is simple, one can always find an open manifold (U, g) such that $(M, g) \subset\subset (U, g)$ where $(\overline{U}, g)$ is simple. We will always understand that (M, g) and (U, g) are related in this way.

Intuitively, a manifold is simple if the boundary is strictly convex and if the whole manifold can be parametrized by geodesic rays starting from any fixed point. The last property can be thought of as an analog for the parametrization $y = s\omega^\perp + t\omega$ of $\mathbb{R}^2$ used in the discussion of the Radon transform in the plane. These parametrizations can be used to prove the analog of the first part of Proposition 5.8 on a simple manifold.

PROPOSITION 5.9. *(Normal operator) If (M, g) is a simple manifold, then $\tilde{I}^*\tilde{I}$ is an elliptic pseudodifferential operator of order -1 in U where $\tilde{I}$ is the geodesic ray transform in $(\overline{U}, g)$.*

It is well known that elliptic pseudodifferential operators can be inverted up to smoothing (and thus compact) operators. This implies an inversion formula as in Proposition 5.8 which however contains a compact error term (resulting in a Fredholm problem). If g is *real-analytic* in addition to being simple then this error term can be removed by the methods of analytic microlocal analysis, thus proving injectivity of I in this case.

For general simple metrics one does not obtain injectivity in this way, but invertibility up to a compact operator implies considerable stability properties for this problem. In particular, if I is known to be injective in (M, g), then suitable small perturbations of I are also injective: it follows from the results of [**7**] that injectivity of I implies the injectivity of the attenuated transform in Section 5.1 for sufficiently small λ. Thus, it remains to prove in some way the injectivity of the unattenuated transform I on simple manifolds.

Energy estimates. The most general known method for proving injectivity of the geodesic ray transform, in the absence of symmetries or real-analyticity, is based on energy estimates. Typically these estimates allow to bound some norm of a function u by some norm of Pu where P is a differential operator, or to prove the uniqueness result that $u = 0$ whenever $Pu = 0$. Such estimates are often proved by integration by parts.

Motivation. Let us consider a very simple energy estimate for the Laplace operator in a bounded open set $\Omega \subseteq \mathbb{R}^2$ with smooth boundary. Suppose that $u \in C^2(\overline{\Omega})$ and $-\Delta u = 0$ in Ω, $u|_{\partial\Omega} = 0$. We wish to show that $u = 0$. To do this, we integrate the equation $-\Delta u = 0$ against the test function u and use the Gauss-Green formula:

$$0 = \int_\Omega (-\Delta u) u \, dx = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu} u \, dS + \int_\Omega |\nabla u|^2 \, dx.$$

Since $u|_{\partial\Omega} = 0$ it follows that $\int_\Omega |\nabla u|^2 \, dx = 0$, showing that u is constant on each component and consequently $u = 0$.

We will now proceed to prove an energy estimate for the geodesic ray transform in the case $(M, g) = (\overline{\Omega}, e)$ where $\Omega \subseteq \mathbb{R}^2$ is a bounded open set with strictly convex boundary and e is the Euclidean metric. This will give an alternative proof of the injectivity result for the Radon transform in $\mathbb{R}^2$, the point being that this proof only uses integration by parts and can be generalized to other geometries.

Suppose $f \in C_c^\infty(M^{\mathrm{int}})$ and $If \equiv 0$. The first step is to relate the integral operator I to a differential operator. This is the standard reduction of the integral geometry problem to a transport equation. We identify SM with $M \times S^1$ and vectors $\omega_\theta = (\cos\theta, \sin\theta) \in S^1$ with the angle $\theta \in [0, 2\pi)$. Consider the function u defined as the integral of f over lines,

$$u(x, \theta) := \int_0^{\tau(x,\theta)} f(x + t\omega_\theta) \, dt, \quad x \in M, \ \theta \in [0, 2\pi).$$

The *geodesic vector field* is the differential operator on SM defined for $v \in C^\infty(SM)$ by

$$\mathscr{H} v(x, \theta) := \frac{\partial}{\partial s} v(x + s\omega_\theta, \theta)\Big|_{s=0} = \omega_\theta \cdot \nabla_x v(x, \theta).$$

Since u is the integral of f over lines and $\mathscr{H}$ differentiates along lines, it is not surprising that

$$\begin{aligned} \mathscr{H} u(x, \theta) &= \frac{\partial}{\partial s} \int_0^{\tau(x,\theta)-s} f(x + (s+t)\omega_\theta) \, dt \Big|_{s=0} \\ &= \int_0^{\tau(x,\theta)} \frac{\partial}{\partial t} f(x + t\omega_\theta) \, dt = -f(x). \end{aligned}$$

Here we used the rule for differentiating under the integral sign.

Thus, if $f \in C_c^\infty(M^{\mathrm{int}})$ and $If \equiv 0$, then u as defined above is a smooth function in SM and satisfies the following boundary value problem for the transport equation involving $\mathscr{H}$:

$$\begin{cases} \mathscr{H}u = -f & \text{in } SM, \\ u = 0 & \text{on } \partial(SM). \end{cases} \tag{5.6}$$

Further, since f does not depend on θ, we can take the derivative in θ and obtain

$$\begin{cases} \partial_\theta \mathscr{H}u = 0 & \text{in } SM, \\ u = 0 & \text{on } \partial(SM). \end{cases} \tag{5.7}$$

We will prove an energy estimate which shows that any smooth solution u of this problem must be identically zero. By (5.6) this will imply that $f \equiv 0$, proving that I is injective (at least on smooth compactly supported functions, which we assume for simplicity).

To establish the energy estimate, we use $\partial_\theta \mathscr{H}u$ as a test function and integrate (5.7) against this function, and then apply integration by parts to identify some positive terms and to show that some terms are zero. This will make use of the following special identity.

Proposition 5.10. *(Pestov identity in $\mathbb{R}^2$) For smooth $u = u(x,\theta)$, one has the identity*

$$|\partial_\theta \mathscr{H}u|^2 = |\mathscr{H}\partial_\theta u|^2 + \mathrm{div}_h(V) + \mathrm{div}_v(W)$$

where for smooth $X = (X^1(x,\theta), X^2(x,\theta))$, the horizontal and vertical divergence are defined by

$$\begin{aligned} \mathrm{div}_h(X) &:= \nabla_x \cdot X(x,\theta), \\ \mathrm{div}_v(X) &:= \nabla_\xi \cdot (X(x, \frac{\xi}{|\xi|}))|_{\xi=\omega_\theta} = \omega_\theta^\perp \cdot \partial_\theta X(x,\theta) \end{aligned}$$

and the vector fields V and W are given by

$$\begin{aligned} V &:= \left[(\omega_\theta^\perp \cdot \nabla_x u)\omega_\theta - (\omega_\theta \cdot \nabla_x u)\omega_\theta^\perp\right] \partial_\theta u, \\ W &:= (\omega_\theta \cdot \nabla_x u)\nabla_x u. \end{aligned}$$

Once the identity is known, the proof is in fact a direct computation and is left as an exercise. Let us now show how the Pestov identity can be used to prove that the only solution to (5.7) is the zero function. Note how the divergence terms are converted to boundary terms by integration by parts, and how one term vanishes because of the boundary condition and the other term is nonnegative.

PROPOSITION 5.11. *If $u \in C^\infty(SM)$ solves* (5.7)*, then $u \equiv 0$.*

PROOF. As promised, we integrate (5.7) against the test function $\partial_\theta \mathscr{H} u$ and use the Pestov identity:

$$\begin{aligned} 0 &= \int_M \int_{S^1} |\partial_\theta \mathscr{H} u|^2 \, d\theta \, dx \\ &= \int_M \int_{S^1} \left(|\mathscr{H} \partial_\theta u|^2 + \mathrm{div}_h(V) + \mathrm{div}_v(W) \right) \, d\theta \, dx. \end{aligned}$$

Here

$$\int_M \mathrm{div}_h(V) \, dx = \int_M \nabla_x \cdot V(x,\theta) \, dx = \int_{\partial M} \nu \cdot V(x,\theta) \, dS(x) = 0$$

since $V(x,\theta) = [\,\cdot\,]\partial_\theta u(x,\theta) = 0$ for $x \in \partial M$ by the boundary condition for u. Also, integrating by parts on S^1,

$$\begin{aligned} \int_{S^1} \mathrm{div}_v(W) \, d\theta &= \int_{S^1} \omega_\theta^\perp \cdot \partial_\theta W \, d\theta = -\int_{S^1} \partial_\theta(-\sin\theta, \cos\theta) \cdot W \, d\theta \\ &= \int_{S^1} \omega_\theta \cdot W \, d\theta = \int_{S^1} |\omega_\theta \cdot \nabla_x u|^2 \, d\theta. \end{aligned}$$

This shows that

$$\int_M \int_{S^1} \left(|\mathscr{H} \partial_\theta u|^2 + |\omega_\theta \cdot \nabla_x u|^2 \right) \, d\theta \, dx = 0.$$

Since the integrand is nonnegative, we see that $\omega_\theta \cdot \nabla_x u = 0$ on SM. Thus $u(\,\cdot\,,\theta)$ is constant along lines with direction ω_θ, and the boundary condition implies that $u = 0$ as required. $\square$

This concludes the energy estimate proof of the injectivity of the ray transform in bounded domains in $\mathbb{R}^2$. A similar elementary argument can be used to show that the geodesic ray transform is injective on simple domains in $\mathbb{R}^2$, see [**1**] or [**18**].

Let us finish by sketching the proof of the injectivity result for the geodesic ray transform on simple manifolds of any dimension $n \geq 2$. For details see [**18**] and [**4**, Section 7] in particular.

PROPOSITION 5.12. *(Injectivity of the geodesic ray transform) Let (M,g) be a simple n-manifold, let $f \in C_c^\infty(M^{int})$, and suppose that $If \equiv 0$. Then $f \equiv 0$.*

PROOF. (Sketch) If (M,g) and f are as in the statement, then as in the $\mathbb{R}^2$ case we define a function $u \in C^\infty(SM)$ by

$$u(x,\xi) := \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,dt, \quad (x,\xi) \in SM.$$

The geodesic vector field acting on smooth functions $v \in C^\infty(SM)$ is given by

$$\mathscr{H} v(x,\xi) := \frac{\partial}{\partial t} v(\gamma(t,x,\xi),\dot\gamma(t,x,\xi))\Big|_{t=0}.$$

Since $If \equiv 0$, we obtain as above that u solves the transport equation

$$\left\{ \begin{array}{ll} \mathscr{H} u = -f & \text{in } SM, \\ u = 0 & \text{on } \partial(SM). \end{array} \right. \tag{5.8}$$

At this point we would like to differentiate the equation in the angular variable ξ to remove the f term. To do this, we need to introduce the horizontal and vertical gradients ∇ and ∂, which are invariantly defined differential operators on so called *semibasic tensors* on SM. For smooth functions $v \in C^\infty(SM)$, they are defined by

$$\begin{aligned} \nabla_j u(x,\xi) &:= \frac{\partial}{\partial x_j}(u(x,\xi/|\xi|)) - \Gamma^l_{jk}\xi^k \partial_l u(x,\xi), \\ \partial_j u(x,\xi) &:= \frac{\partial}{\partial \xi_j}(u(x,\xi/|\xi|)). \end{aligned}$$

The geodesic vector field can be defined on semibasic tensor fields via $\mathscr{H} := \xi^j \nabla_j$. We also define $|\partial v|^2 := g^{jk}\partial_j v \partial_k v$, etc. One then has the following general Pestov identity whose proof is again a direct computation (which uses basic properties of ∇ and ∂). A major difference to the Euclidean case is the appearance of a curvature term.

PROPOSITION 5.13. *(Pestov identity) If (M,g) is an n-manifold and $u \in C^\infty(SM)$, one has the identity*

$$|\partial \mathscr{H} u|^2 = |\mathscr{H} \partial u|^2 + \operatorname{div}_h(V) + \operatorname{div}_v(W) - R(\partial u, \xi, \xi, \partial u)$$

where the horizontal and vertical divergence are defined by

$$\operatorname{div}_h(X) := \nabla_j X^j, \quad \operatorname{div}_v(X) := \partial_j X^j,$$

and V and W are given by

$$V^j := \langle \partial u, \nabla u\rangle \xi^j - (\mathscr{H} u)\partial^j u, \quad W^j := (\mathscr{H} u)\nabla^j u.$$

Also, R is the Riemann curvature tensor.

We now take the vertical gradient in (5.8) and obtain

$$\begin{cases} \partial\mathscr{H}u = 0 & \text{in } SM, \\ u = 0 & \text{on } \partial(SM). \end{cases} \tag{5.9}$$

Similarly as in the $\mathbb{R}^2$ case, we pair this equation against $\partial\mathscr{H}u$, integrate over SM and use the Pestov identity to obtain that

$$\int_{SM} \left[|\mathscr{H}\partial u|^2 + \mathrm{div}_h(V) + \mathrm{div}_v(W) - R(\partial u, \xi, \xi, \partial u) \right] \, d(SM) = 0.$$

Integrating by parts, the $\mathrm{div}_h(V)$ term vanishes and the $\mathrm{div}_v(W)$ term gives a positive contribution as in the Euclidean case. One eventually gets that

$$\int_{SM} \left[|\mathscr{H}\partial u|^2 - R(\partial u, \xi, \xi, \partial u) \right] \, d(SM) + (n-1) \int_{SM} |\mathscr{H}u|^2 \, d(SM) = 0.$$

The first term is related to the *index form* for a geodesic $\gamma = \gamma(\,\cdot\,, x, \xi)$ in (M, g), which is given by

$$I(X, X) := \int_0^{\tau(x,\xi)} (|D_{\dot\gamma} X|^2 - R(X, \dot\gamma, \dot\gamma, X)) \, dt$$

for vector fields X on γ with $X(0) = X(\tau(x, \xi)) = 0$. If (M, g) is simple, or more generally if no geodesic in (M, g) has conjugate points, then the index form is known to be always nonnegative. This implies that the first term above is nonnegative, showing that $\mathscr{H}u = 0$ and $u = 0$ as required. From (5.8) one obtains that $f \equiv 0$. $\square$

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