A Short Course on the Hilbert Space Theory of Evolutionary Equations.

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Exercises

Please Submit Solutions to
EITHER exercises 5,7,9,10
OR any TWO exercises of 13-20

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Exercise Section I

Exercise 1. Consider the following function space:

$$L^2(S_{\mathbb{C}}(0,1),(2\pi iz)^{-1}\mathrm{d}z) := \{f : S_{\mathbb{C}}(0,1) \to \mathbb{C} \,|\, \int_{S_{\mathbb{C}}(0,1)} |f(z)|^2 \frac{1}{2\pi iz}\ \mathrm{d}z < \infty\},$$

where $S_{\mathbb{C}}(0,1) := \{z \in \mathbb{C} \,|\, |z| = 1\}$. Show that

$$\begin{aligned}\langle .|.\rangle : L^2(S_{\mathbb{C}}(0,1),(2\pi iz)^{-1}\mathrm{d}z) \times L^2(S_{\mathbb{C}}(0,1),(2\pi iz)^{-1}\mathrm{d}z) &\to \mathbb{C}\\ (f,g) &\mapsto \int_{S_{\mathbb{C}}(0,1)} \overline{f(z)}\, g(z) \frac{1}{2\pi iz}\ \mathrm{d}z\end{aligned}$$

defines an inner product on $L^2(S_{\mathbb{C}}(0,1),(2\pi iz)^{-1}\mathrm{d}z)$.

Exercise 2. Find an isometric linear bijection (i.e. a unitary mapping) $U : L^2\left(\left]-\frac{1}{2},-\frac{1}{2}\right[,\mathbb{C}\right) \to L^2(S_{\mathbb{C}}(0,1),(2\pi iz)^{-1}\mathrm{d}z)$. Show that the functions $S_{\mathbb{C}}(0,1) \ni z \mapsto z^k$ form an orthonormal set in $L^2(S_{\mathbb{C}}(0,1),(2\pi iz)^{-1}\mathrm{d}z)$. What are the (via U) corresponding functions in $L^2\left(\left]-\frac{1}{2},-\frac{1}{2}\right[,\mathbb{C}\right)$?

Exercise 3. Let (M,d_M) and (N,d_N) be metric spaces, where N is a complete metric space. A mapping $f : D(f) \subseteq M \to N$ is called Cauchy continuous near $a \in M$ if

$$\bigwedge_{\varepsilon\in]0,\infty[}\ \bigvee_{\delta\in]0,\infty[}\ \bigwedge_{u,v\in B_M(a,\delta)\cap D(f)} d_N\left(f\left(u\right),f\left(v\right)\right) < \varepsilon.$$

The mapping f is called Cauchy continuous, if it is Cauchy continuous at all points of $\overline{M}$. Show that if f is Cauchy continuous then it has a unique continous extension $\overline{f}$ defined on $\overline{D\left(f\right)}$.

Exercise 4. The complex numbers $\mathbb{C} = \left\{\begin{pmatrix} x & -y \\ y & x\end{pmatrix} \,\middle|\, x,y \in \mathbb{R}\right\}$ are a Hilbert space over $\mathbb{C}$ with inner product

$$(\alpha,\beta) \mapsto \alpha^{\top}\beta.$$

Consider the complex numbers $\mathbb{C} = \left\{\begin{pmatrix} x & -y \\ y & x\end{pmatrix} \,\middle|\, x,y \in \mathbb{R}\right\}$ as a linear space $\mathbb{C}_{\mathbb{R}}$ over the field $\mathbb{R}$ (here identified with $\left\{\begin{pmatrix} x & 0 \\ 0 & x\end{pmatrix} \,\middle|\, x \in \mathbb{R}\right\}$ and construct its complexification. Provide an orthonormal basis for this complexification.

Exercise Section II

EXERCISE 5. (HW) Let H_0, H_1 be complex (or real) Hilbert spaces and $A : D(A) \subseteq H_0 \to H_1$ a linear operator.

- Prove that the following statements are equivalent:
  (1) A is closed,
  (2) for all sequences $(x_n)_{n\in\mathbb{N}} \in D(A)^{\mathbb{N}}$ with $x_n \overset{n\to\infty}{\to} x \in H_0$ and $Ax_n \overset{n\to\infty}{\to} y \in H_1$ we have that $x \in D(A)$ and $y = Ax$.
- Prove that the following statements are equivalent:
  (1) A is a closable operator,
  (2) there is a closed, linear operator $B : D(B) : H_0 \to H_1$ such that $A \subseteq B$,
  (3) for all sequences $(x_n)_{n\in\mathbb{N}} \in D(A)^{\mathbb{N}}$ with $x_n \overset{n\to\infty}{\to} 0 \in H_0$ and $Ax_n \overset{n\to\infty}{\to} y \in H_1$ we have that $y = 0$.

EXERCISE 6. Let H_0, H_1 be complex Hilbert spaces and $A : D(A) \subseteq H_0 \to H_1$ a closed, linear operator. Show that $D(A)$ equipped with the graph inner product

$$\begin{aligned} \langle \cdot|\cdot \rangle_{D(A)} : D(A) \times D(A) &\to \mathbb{C} \\ (x, y) &\mapsto \langle x|y \rangle_{H_0} + \langle Ax|Ay \rangle_{H_1} \end{aligned}$$

is a (complex) Hilbert space.

EXERCISE 7. (HW) Let H_0, H_1, H_2, H_3 be complex Hilbert spaces and $A : D(A) \subseteq H_0 \to H_1$ a closed, linear operator. Prove the following

(1) if $B : D(B) \subseteq H_1 \to H_2$ is a densely defined linear operator and such that $D(BA)$ is dense in H_0, then
$$A^* B^* \subseteq (BA)^*,$$
(2) if $B : H_1 \to H_2$ is a continuous, densely defined linear operator and such that $D(BA)$ is dense in H_0, then
$$A^* B^* = (BA)^*,$$
(3) if $U : H_1 \to H_2$, $V : H_3 \to H_0$ are continuous linear operators, V a bijection, then
$$V^* A^* U^* = (UAV)^*.$$

Exercise Section III

Exercise 8. Consider ∂_0 as the closure of

$$\begin{aligned}\mathring{C}_1(\mathbb{R},\mathbb{C}) \subseteq H_{\nu,0}(\mathbb{R}) &\to H_{\nu,0}(\mathbb{R})\\ \varphi &\mapsto \varphi',\end{aligned}$$

where

$$H_{\nu,0}(\mathbb{R}) := \left\{\varphi \in L^{2,\mathrm{loc}}(\mathbb{R},\mathbb{C}) \,\middle|\, \int_{\mathbb{R}} |\varphi(t)|^2 \exp(-2\nu t)\ dt < \infty\right\},\ \nu \in \left]0,\infty\right[,$$

is a Hilbert space with inner product

$$\langle u|v\rangle_{\nu,0} = \int_{\mathbb{R}} \overline{u(t)}\, v(t) \exp(-2\nu t)\ dt.$$

(1) Show that the elements of the Hilbert space
$$H_{\nu,1}(\mathbb{R}) := D\left((\partial_0 - \nu)^*\right),$$
equipped with the graph inner product, can be approximated in $H_{\nu,1}(\mathbb{R})$ by sequences $(\varphi_k)_{k\in\mathbb{N}}$ in $H_{\nu,1}(\mathbb{R})$ such that φ_k has compact support, i.e. vanishes outside bounded sets, $k \in \mathbb{N}$.
(2) Show that the elements of $H_{\nu,1}(\mathbb{R})$ can be approximated in $H_{\nu,1}(\mathbb{R})$ by sequences $(\varphi_k)_{k\in\mathbb{N}}$ in $\mathring{C}_1(\mathbb{R},\mathbb{C})$.
(3) Prove that the domain of ∂_0 is $H_{\nu,1}(\mathbb{R})$ and that $(\partial_0 - \nu)$ is skew-selfadjoint, i.e. $(\partial_0 - \nu)^* = -(\partial_0 - \nu)$.
(4) Prove that ∂_0 is strictly positive definite in the space $H_{\nu,0}(\mathbb{R})$ considered as a real Hilbert space.

Exercise 9. (HW) Let $A : H_0 \to H_0$ be a continuous linear bijection with

$$\mathfrak{Re}\,\langle x|Ax\rangle_{H_0} \geq c_0\, |x|^2_{H_0}$$

for some $c_0 \in \left]0,\infty\right[$ and all $x \in H_0$. Consider a closed subspace $U \subseteq H_0$ and its canonical embedding

$$\begin{aligned}\iota_U : U &\to H_0,\\ x &\mapsto x.\end{aligned}$$

Show that

$$\iota_U^* A \iota_U : U \to U$$

is a bijection with

$$\mathfrak{Re}\,\langle x|\iota_U^* A \iota_U x\rangle_{H_0} \geq c_0\, |x|^2_{H_0}$$

for all $x \in U$.

Exercise 10. (HW) Let $A : H_0 \to H_0$ be a continuous linear bijection. Consider a closed subspace $U \subseteq H_0$ and its canonical embedding

$$\begin{aligned}\iota_U : U &\to H_0,\\ x &\mapsto x.\end{aligned}$$

Show by an explicit example that

$$\iota_U^* A \iota_U : U \to U$$

need not always be a bijection.

Exercise 11. Consider the equation

$$-\partial\beta\mathring{\partial}u = f$$

on the interval $\left]-1,+1\right[$, where β is a multiplication operator given by the function

$$\beta(x) = \begin{cases}\alpha_+ & \text{for } x \geq 0,\\ -\alpha_- & \text{for } x < 0,\end{cases}$$

where $\alpha_\pm \in \left]0,\infty\right[$. Characterize the possible choices of such β for which well-posedness in $L^2\left(\left]-1,+1\right[\right)$ holds?

EXERCISE 12. Let $P_U : H \to H$ be the orthogonal projector onto the closed, linear subspace U of the Hilbert space H. Show that with the canonical embedding

$$\begin{aligned} \iota_U : U &\to H, \\ x &\mapsto x, \end{aligned}$$

we get

$$P = \iota_U \iota_U^* = |\iota_U^*| \,.$$

Exercise Section IV

Exercise 13. (HW) Consider

$$\overline{(\partial_0 P + (1-P) + \partial_1)}u = f$$

in $H_{\nu,0}\left(\mathbb{R}, L^2(\mathbb{R},\mathbb{C})\right)$, where P is an orthogonal projector in $L^2(\mathbb{R},\mathbb{C})$. Why is this problem well-posed? Give an integral representation of the solution for an $f \in \mathring{C}_\infty(]-1,+1[,\mathbb{C})$ under the assumption that P commutes with ∂_1.

Hint: Use that the solution of $(\partial_0 + \partial_1)\, u = f$ is given by

$$u(t,x) = \int_{-\infty}^{x} f(t-x+r, r)\; dr.$$

Exercise 14. (HW) Consider

$$\partial_0 \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_1 \end{pmatrix} + \begin{pmatrix} (1-\varepsilon_0) & 0 \\ 0 & (1-\varepsilon_1) \end{pmatrix} + \begin{pmatrix} 0 & \partial_1 \\ \mathring{\partial}_1 & 0 \end{pmatrix}$$

in $H_{\nu,0}\left(\mathbb{R}, L^2(]-1,+1[,\mathbb{C})\right)$ and discuss the four cases for $\varepsilon_0, \varepsilon_1 \in \{0,1\}$ in correspondence to the related second order problems.

Exercise 15. (HW) Consider

$$\partial_0 + \begin{pmatrix} 0 & \partial_1^{(-1)} \\ \partial_1^{(1)} & 0 \end{pmatrix}$$

in $H_{\nu,0}\left(\mathbb{R}, \iota_e^*\left[L^2(\mathbb{R})\right] \oplus \iota_o^*\left[L^2(\mathbb{R})\right]\right)$. Here

$$\partial_1^{(1)} := \overline{\iota_o^* \partial_1}\iota_e,$$
$$\partial_1^{(-1)} := -\left(\partial_1^{(1)}\right)^*$$

and ι_e, ι_o are the canonical embeddings of the even and odd functions in $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$, respectively. Show

$$\partial_1^{(-1)} = \overline{\iota_e^* \partial_1}\iota_o$$

and unitary congruence to $\partial_0 + \partial_1$ in $H_{\nu,0}\left(\mathbb{R}, L^2(\mathbb{R})\right)$.

Exercise 16. (HW) Show that $D(\partial_0) \cap D(A)$ is dense in the domain of $(\partial_0 M_0 + M_1 + A)^*$.

Exercise Section V

Exercise 17. (HW) In the theory of linear heat conduction the entropy η is linked to the heat flux q via

$$T_0 \partial_0 \left(\varrho_0 \eta\right) = -\operatorname{div} q + h_0 \,,$$

where h_0 denotes an external heat source, ϱ_0 is the coefficient of mass density and $T_0 \in \left]0,\infty\right[$ a reference temperature. Entropy and temperature are coupled via a material law of the form:

$$\varrho_0 \eta = \nu \theta.$$

The heat flux also depends on the temperature according to Fourier's law

$$q = -\kappa \operatorname{grad} \theta,$$

where κ is a coefficient of heat conduction. Already Maxwell suggested in 1867, later re-iterated by Cattaneo (1958) and Vernotte (1958), to replace Fourier's law by

$$\tau_0 \partial_0 q + q = -\kappa \operatorname{grad} \theta,$$

where $\tau_0 \in \left]0,\infty\right[$.

Can you develop a model system with the canonical form $\partial_0 M_0 + M_1 + A$ for this so-called Maxwell-Cattaneo-Vernotte model (MCV model)?

Exercise 18. (HW) Consider the formal system

$$\begin{pmatrix} \left(\tau_0 \kappa^{-1} \partial_0 + \kappa^{-1}\right) & \begin{pmatrix} \operatorname{grad} & -\operatorname{div} \end{pmatrix} \\ \begin{pmatrix} \operatorname{div} \\ -\operatorname{grad} \end{pmatrix} & \begin{pmatrix} \varrho c \partial_0 & 0 \\ 0 & C^{-1} \partial_0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} q \\ \theta \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}.$$

In the homogeneous, isotropic case, C has the simple form

$$C := \alpha_0 \operatorname{sym}_0 + \alpha_1 \mathbb{P} + \alpha_2 \operatorname{skew},\ \alpha_0, \alpha_1,\ \alpha_2 \in \left]0,\infty\right[,$$

where

$$\begin{aligned} \mathbb{P} &:= \frac{1}{3} \operatorname{trace}^* \operatorname{trace}, \\ \operatorname{sym} \sigma &:= \frac{1}{2} \left(\sigma + \sigma^\top\right), \\ \operatorname{sym}_0 &:= (1 - \mathbb{P}) \operatorname{sym} = \operatorname{sym} (1 - \mathbb{P}), \\ \operatorname{skew} \sigma &:= \frac{1}{2} \left(\sigma - \sigma^\top\right). \end{aligned}$$

with

$$\operatorname{sym} : \mathbb{C}^{3\times 3} \to \mathbb{C}^{3\times 3}$$

and

$$\begin{aligned} \operatorname{trace} : \mathbb{C}^{3\times 3} &\to \mathbb{C}, \\ \sigma &\mapsto \sum_{i=1}^{3} \sigma_{ii}. \end{aligned}$$

It is

$$\begin{aligned} \operatorname{trace}^* : \mathbb{C} &\to \mathbb{C}^{3\times 3}, \\ z &\mapsto z \mathbb{I}_{3\times 3}, \end{aligned}$$

where $\mathbb{I}_{3\times 3}$ denotes the identity matrix in $\mathbb{C}^{3\times 3}$. Reformulate this in the isotropic case as a system for θ and q alone assuming that all coefficients are constant. Compare with the original Guyer-Krumhansl model

$$\begin{aligned} \left(1 + \tau_0 \partial_0\right) q &= -\kappa \operatorname{grad} \theta + \mu_1 \Delta q + \mu_2 \operatorname{grad} \operatorname{div} q \\ \varrho c \partial_0 \theta &= -\operatorname{div} q + h \end{aligned}$$

and determine parameter ranges for well-posedness (assuming a skew-selfadjoint realization of $\begin{pmatrix} (0) & \begin{pmatrix} \operatorname{grad} & -\operatorname{div} \end{pmatrix} \\ \begin{pmatrix} \operatorname{div} \\ -\operatorname{grad} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$ in the form $\begin{pmatrix} (0) & -C^* \\ \overline{C} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$ with $C \subseteq \begin{pmatrix} \operatorname{div} \\ -\operatorname{grad} \end{pmatrix}$ and a choice of dense domain).

Exercise 19. (HW) The classical Schrdinger operator is of the form

$$\partial_0 - \mathrm{i} \operatorname{div} \mathring{\operatorname{grad}} + V(\mathbf{m}) = \partial_0 + \mathrm{i} \left|\mathring{\operatorname{grad}}\right|^2 + V(\mathbf{m})$$

where the so-called potential $V(\mathbf{m})$ is a suitable multiplication operator. The so called relativistic Schrdinger operator replaces $\left|\mathring{\operatorname{grad}}\right|^2$ by $\left|\mathring{\operatorname{grad}}\right|$:

$$\partial_0 + \mathrm{i} \left|\mathring{\operatorname{grad}}\right| + V(\mathbf{m}).$$

By separating real and imaginary part we obtain a system of our standard first order form

$$\partial_0 + M_1 + \begin{pmatrix} 0 & \operatorname{div} \mathring{\operatorname{grad}} \\ -\operatorname{div} \mathring{\operatorname{grad}} & 0 \end{pmatrix} = \partial_0 + M_1 + \begin{pmatrix} 0 & -\left|\mathring{\operatorname{grad}}\right|^2 \\ \left|\mathring{\operatorname{grad}}\right|^2 & 0 \end{pmatrix},$$

where

$$M_1 := \begin{pmatrix} \mathfrak{Re}\, V(\mathbf{m}) & -\mathfrak{Im}\, V(\mathbf{m}) \\ \mathfrak{Im}\, V(\mathbf{m}) & \mathfrak{Re}\, V(\mathbf{m}) \end{pmatrix}.$$

In the case of the relativistic Schrdinger operator this procedure leads to

$$\partial_0 + M_1 + \begin{pmatrix} 0 & -\left|\mathring{\operatorname{grad}}\right| \\ \left|\mathring{\operatorname{grad}}\right| & 0 \end{pmatrix}.$$

Show that the latter is unitarily congruent to a system operator of the form

$$\partial_0 + \widetilde{M_1} + \begin{pmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix}$$

considered in $H_{\nu,0}\left(\mathbb{R}, L^2(\Omega) \oplus \overline{\mathring{\operatorname{grad}}\left[D\left(\mathring{\operatorname{grad}}\right)\right]}\right)$. What is $\widetilde{M_1}$?

Exercise 20. (HW) Consider a system of the form

$$\left(\partial_0 M_0 + M_1 + \partial_0^{-1} M_2 + A\right) U = F,$$

where M_0, M_2 are non-negative, selfadjoint and A, M_1 are skew-selfadjoint. Using that for $U \in H_{\nu,1}(\mathbb{R}, H)$ we have

$$\partial_0 |U|_0^2 = 2\, \mathfrak{Re} \langle U | \partial_0 U \rangle_0,$$

where the derivative on the left-hand side is in the sense of distributions[1],

show that formally "energy conservation" holds in the sense of

$$\frac{1}{2} \left|\sqrt{M_0} U\right|_0^2 (t) + \frac{1}{2} \int_s^t \left|\sqrt{M_2} \partial_0^{-1} U\right|_0^2 (r)\, dr = \frac{1}{2} \left|\sqrt{M_0} U\right|_0^2 (s)$$

for $s, t \in I$, $s < t$, where I is an open interval in which F vanishes. How can the reasoning be made rigorous?

[1] $|\cdot|_0$ denotes the norm and $\langle \cdot | \cdot \rangle_0$ the inner product of H.