Home assignments

Pick up at least four (4) of the following exercises. Exercise 1 (the essay) is mandatory for all. The other three exercises can be freely chosen from the list below. Please send your answers by email to the lecturer (**Aku.Seppanen@uef.fi**) in the form of a short report (use pdf-format). The deadline for sending the answers is September 7th, 2014. For each answer (except for the essay), include a couple of illustrating figures and a few lines of explanation to the report. The language in the answers can be English or Finnish. Although the minimum requirement is four exercises, I encourage you to carry out as many exercises as possible.

1. *Essay.* Write a short (min 250 words) essay on "Ill-posed inverse problems". The title can also be something else, depending on your own interest – as long as it is related to inverse problems, e.g. "Computational inverse problems", "Bayesian inverse problems", or (and this is even better) something related to your own research, such as: "Inverse problems in industrial process monitoring" or "Inverse problems in stem cell research".

2. *Linear LS-problem.* Let $x \in [0,\ 2\pi]$ be a controllable variable, and the model between x and an observable variable y of the form
$$y = a\sin(x) + b\cos(2x)$$
where a and b are unknown parameters. Simulate noisy measurements $\{y_i,\ i = 1, \ldots, 20\}$ corresponding to a set of points $\{x_1 = 0,\ x_2 = \pi/10,\ \ldots,\ x_{20} = 2\pi\}$, true parameter values $a_{\text{true}} = 1,\ b_{\text{true}} = 2$, and using additive Gaussian noise with variance 0.1. Based on the noisy data, estimate a and b using LS-solution. (Hint: cf. Lecture examples 3.2 and 3.3.)

3. *Non-linear LS-problem.* Let $x \in [0,\ 2\pi]$ be a controllable variable, and the model between x and an observable variable y of the form
$$y = a\cos(x + b)$$
where a and b are unknown parameters. Simulate noisy measurements $\{y_i,\ i = 1, \ldots, 20\}$ corresponding to a set of points $\{x_1 = 0,\ x_2 = \pi/10,\ \ldots,\ x_{20} = 2\pi\}$, true parameter values $a_{\text{true}} = 1,\ b_{\text{true}} = \frac{3}{4}\pi$, and using additive Gaussian noise with variance 0.1. Based on the noisy data, estimate a and b using LS-solution. (Hint: cf. Lecture example 3.6.) Try several different choices for the initial guesses for the parameters a and b. Note that in this example, the LS solution is not unique. Why?

4. *Generalized Tikhonov regularization.* **Lecture example 4.1** (numerical differentiation): Try out LS and regularized LS solutions with different discretizations; for example with M=20 and M=200. In the regularized solutions, try various different values of regularization parameter α, perhaps from region $\alpha \in [.2,\ 4]$. Does a feasible choice for α depend on the discretization? How?

5. *Generalized Tikhonov regularization.* Repeat the **Lecture example 4.1** (numerical differentiation) in a case where the function $y = y(x)$ is of the form

$$y(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]$$

where $\sigma = 0.02$, and erf is the *error function* (In Matlab, use command *erf*). The derivative of function $y(x)$ is then

$$q(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

In this example, set $x \in [-1,\ 1]$, and select M=40. Try regularized solutions with various different regularization parameters α. How does the regularized solution work in this example? Why?

6. *Metropolis-Hastings MCMC.* Run **Lecture example 5.2** with different choices of parameter γ. Try for example $\gamma = \{0.02,\ 0.5,\ 2\}$. With different values of γ: 1) How well do the samples represent the posterior density π_{post}? 2) Which percentage of the samples from proposal density are accepted?

7. *Transmission tomography.* In this exercise, use m-file *TransmissionTomographyExample.m* to study some basic features on transmission tomography, such as x-ray CT. You will also need function *ConstructObsMatrixTransTomo.m*. Both m-files are provided for you together with these assignments. First, figure out (at least roughly) what the code does. I've tried to write comments that help following where it goes.

   As the first simulation study, start with the parameter configuration provided in the codes. Here, the discretization of the image for the inverse problem is same as that used for simulating the data. In addition, the observation noise is extremely low. (These two choices signify committing an *inverse crime!*) Furthermore, the number of projections is high, and the projections cover angles from 0 to 180 degrees, i.e. full view. Verify that with these choices even the (unregularized) LS solution works well.

   Next, start making changes to the above listed parameter choices. First, add a bit larger noise to the data. Does the (unregularized) LS solution still work? How about the (standard) Tikhonov regularized solution? Also, to avoid inverse crime, use denser discretization for simulating the data (Choose e.g. Ny1 = 120, and keep Ny = 50). Again see how to two different reconstructions behave when the observations are corrupted by noise (try different noise variances). Also study the effects of projection angles (smaller number of projections and/or limited angle tomography case, etc.). Study the effect of regularization parameter, etc.

8. *Transmission tomography* (continued). You can have even more fun with the transmission tomography simulation code. Try out (instead of standard Tikhonov regularization) generalized Tikhonov regularization with 2D difference matrix (giving numerical derivatives at both x- and y-directions), and/or higher order numerical derivatives in 2D. Or, take Bayesian approach, and formulate a smoothness prior using the squared exponential function discussed in the last lecture.