# Statistical and Computational Inverse Problems with Applications

# Part 2: Introduction to inverse problems and example applications

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Jyväskylä Summer School
August 11-13, 2014

# Forward problem

- The cause is known.
- Task: find out the consequence.

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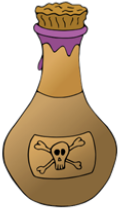

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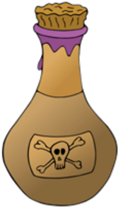

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# The inverse of the forward problem

- The consequence is observed.
- Task: Find out the cause.

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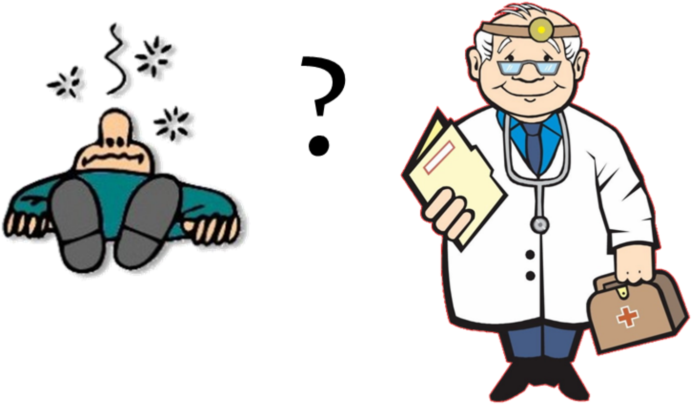
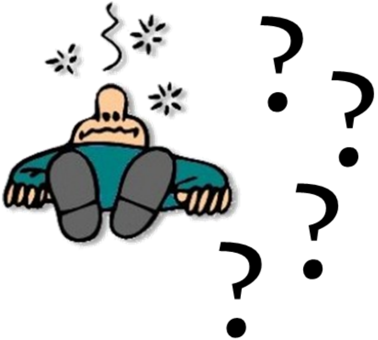


# An ill-posed inverse problem

- If finding out the cause based on the observed consequence is a very challenging problem...



# An ill-posed inverse problem

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# Hadamard conditions

According to Hadamard (1865-1963), a problem is well-posed if the following three conditions hold:

1. A solution exists
2. The solution is unique
3. The solution depends continuously on the input

The third condition is related to stability of the solutions. Solutions of numerical problems can be unstable (intolerant to measurement noise and modeling errors) even if the solution depends continuously on data.

# Example: Differentiation and the choice of computation grid

- 1D function $g(x)$; denote $f(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x}$
- Consider a situation where one observes finite number of samples $\{g(x_i)\}$ (at points $x_1, \ldots, x_N$) such that the observations are corrupted by additive Gaussian noise.
- Thus, the observations are of the form

$$g_k^\delta \doteq g(x_k) + n_k = g_k + n_k \ ,$$

where $\mathbb{E}\{n_k\} = 0$ (mean), $\mathrm{var}(n_k) = \delta^2$ (variance) and $\mathbb{E}\{n_k n_m\} = 0, k \neq m$, i.e., errors $n_k$ are mutually independent.

- Approximate the derivative $f(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x}$ by finite difference approximation $f_\ell^\delta$ in the same (equispaced) grid $y_\ell = x_k$ $(h = x_k - x_{k-1})$. We get

$$\begin{aligned} f_k^\delta &\doteq Dg_k^\delta = D(g_k + n_k) = Dg_k + Dn_k \\ &= \frac{g_k - g_{k-1}}{x_k - x_{k-1}} + \frac{n_k - n_{k-1}}{x_k - x_{k-1}} \, . \end{aligned}$$

- The first term has the property $\to g'(x_k) = f(x_k)$ as $x_k - x_{k-1} \to 0$.
- The second term represents the estimation error for $f(x)$.

- This term is random with variance

$$
\begin{aligned}
\operatorname{var}\left(\frac{n_k - n_{k-1}}{x_k - x_{k-1}}\right) &= \mathbb{E}\left\{\left(\frac{n_k - n_{k-1}}{x_k - x_{k-1}}\right)^2\right\} - \mathbb{E}\left\{\frac{n_k - n_{k-1}}{x_k - x_{k-1}}\right\}^2 \\
&= (x_k - x_{k-1})^{-2}\mathbb{E}\{n_k^2 + n_{k-1}^2 - 2n_k n_{k-1}\} \\
&= (x_k - x_{k-1})^{-2}(\mathbb{E}\{n_k^2\} + \mathbb{E}\{n_{k-1}^2\} \\
&\quad -2\mathbb{E}\{n_k n_{k-1}\}) \\
&= \frac{2\delta^2}{(x_k - x_{k-1})^2} \\
&\to \infty\,, \quad \text{as } x_k - x_{k-1} \to 0\,.
\end{aligned}
$$

- Thus, the variance of the estimation error increases w.r.t the accuracy of the computation grid. Figure 1 shows noisy data $g_h^\delta$, true integral function $g(x)$, true target function $f(x)$ and estimates $f_h^\delta$ with three different mesh parameters $h = x_k - x_{k-1}$.

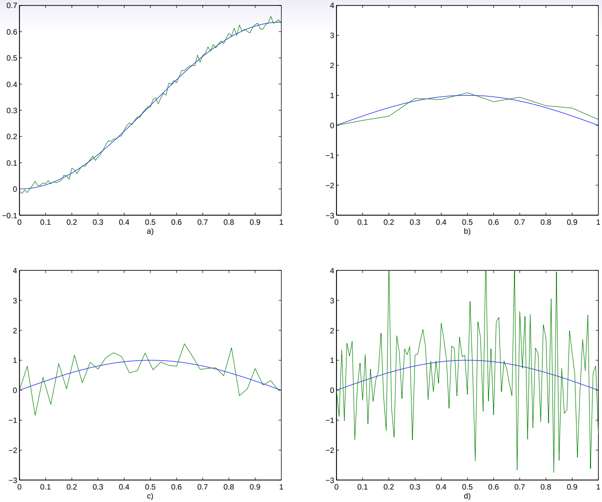



Figure 1: a) Integral function $g(x)$ and noisy observation $g_h^\delta$, b)-d) $f(x)$ and estimates $f_h^\delta$, when $h = 0.1, 0.03, 0.01$, respectively.

- Clearly, the optimal mesh parameter $h$ with respect the estimation error $\|f - f_k^{\delta}\|_2$ depends on the noise level $\delta$. In this example, the regularization of the problem was carried out by tuning the discretization. This is an obsolete, non-recommended approach.

# Estimation problems

- In this course we mostly consider estimation problems based on an observation model of the form

$$g = h(f, n)$$

  where $g$ = measurable variable, $f$ = primary unknow of interest, $n$ = noise, and $h(f, n)$ = numerical model that connects $g$ with the unknown $f$ and $n$.
- Our aim is to estimate $f$ based on observed $g$.
- In cases of ill-posed inverse problems, conventional solutions, such as LS-solutions are non-unique and/or extremely intolerant to measurement noise and modeling errors.
- In deterministic inversion framework, $f$ is considered as deterministic but unknow variable, while in Bayesian (statistical) inversion framework, $f$ is modeled as a random variable.

# Two simple examples

- The following two examples are considered in the lectures. Matlab codes are provided.
- **Example 2.1:** Linear model

$$g = Kf, \quad \text{where } f, g \in \mathbb{R}^2, \ K = \begin{pmatrix} 0 & 1 \\ a & 1 \end{pmatrix}, \quad a \neq 0.$$

  We demonstrate that the estimate for $f$ becomes intolerant to noise in observations $g$, when $a$ gets small.
- **Example 2.2:** In the second example, we consider the tolerance of the estimates in Example 2.1 to modeling errors.

# Notes on the above examples

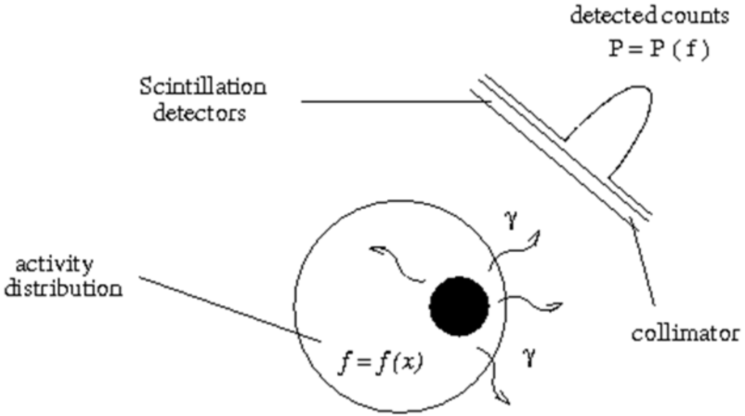
- The solutions in above examples were unique.
- The problems were not *numerically* unstable (Matlab does not complain about inverting the matrix $K$...)

- What caused the instability?

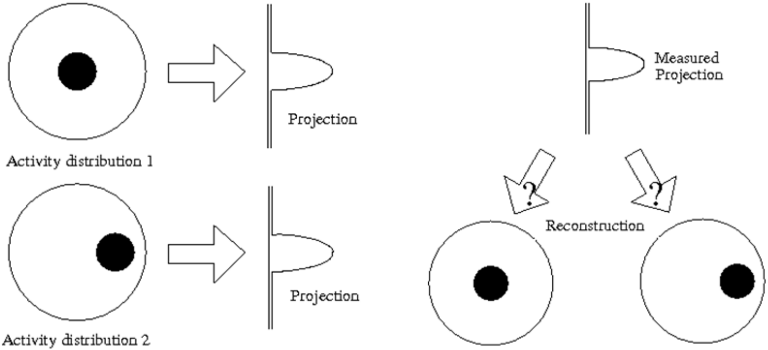
# What caused the instability?

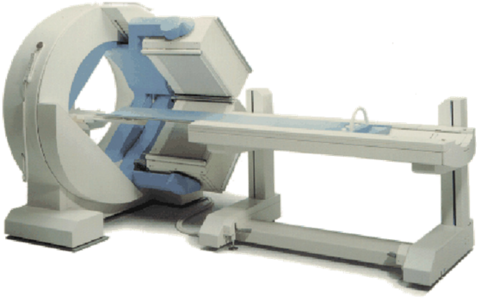
- When $a << 1$, then
$$K \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}$$
is very small.
- Then, the contribution of $f_1$ to the data is very small, i.e., a large change in $f_1$ causes only a small change in the observable variable $g$.
- "Inversely" thinking (and loosely speaking), accommodating to a small change in the observed data (caused by measurement noise or modeling error), requires a large change in $f_1 \Rightarrow$ Instability.
- Even more loosely speaking, vector $(1\ 0)^{\mathrm{T}}$ is "almost" in the null-space of $K$.
- The intolerances with respect to measurement noise and modeling errors are the characterizing features of ill-posed inverse problems.

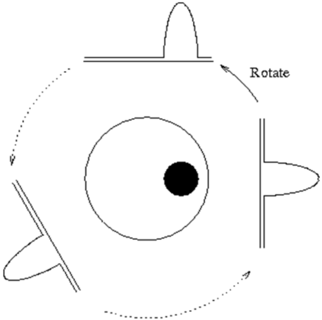
# Single photon emission computed tomography (SPECT)



- If one projection only, non-uniqueness



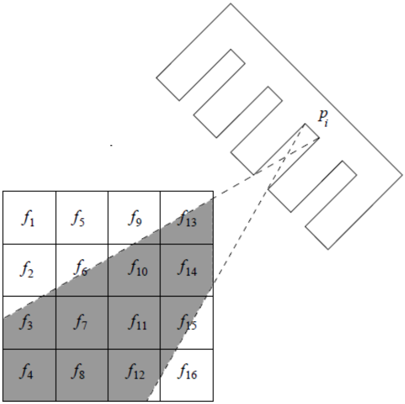



Rotate

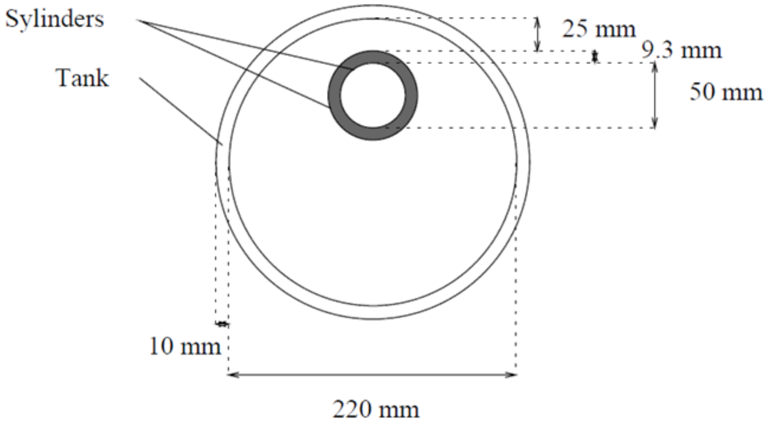
- Linear observation model

$$g = Kf + n$$

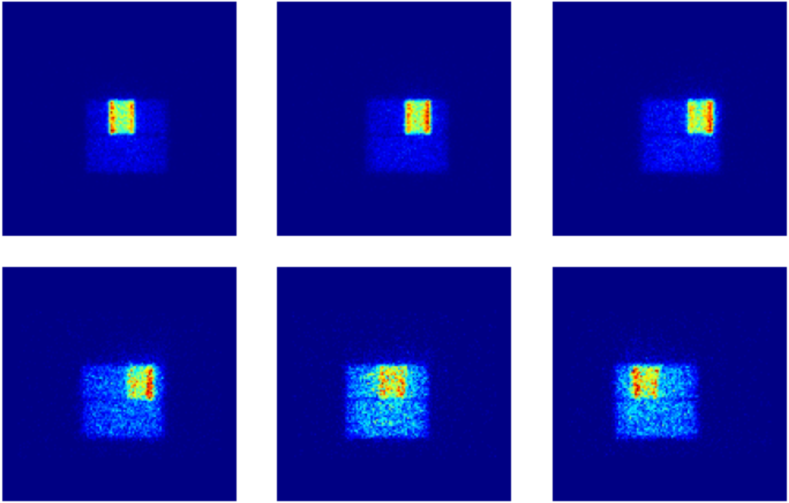
where $g = [P_1, \ldots P_M]^{\mathrm{T}} \in \mathbb{R}^M$, $K \in \mathbb{R}^{N \times M}$, $f = [f_1, \ldots f_M]^{\mathrm{T}} \in \mathbb{R}^M$ and $n \in \mathbb{R}^M$.



# Experiment with a cylindrical phantom

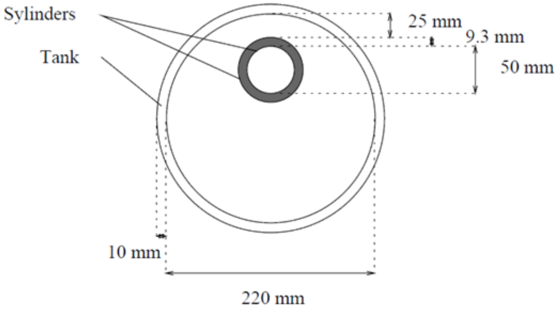


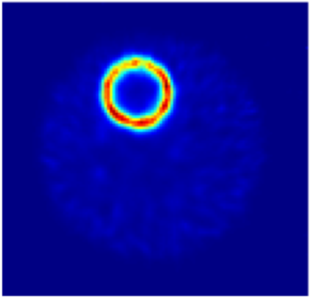
# Projection images at different directions



# Reconstruction

- Reconstructed image on one horizontal plane



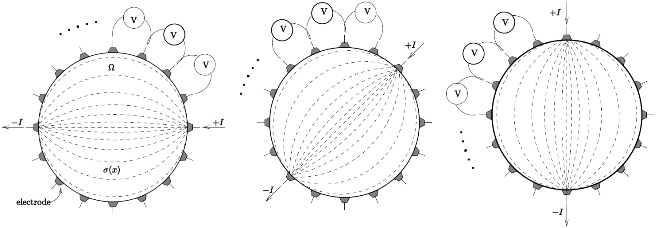
- The ill-posedness of the SPECT inverse problem depends e.g. on the number (and span) of projections, the collimator and the attenuation coefficient of the material

# Inverse problems related to diffusion phenomena

- In physics, it is difficult to deduce the input of a diffusion-type process based on the measured outcome of the process (Kaipio & Somersalo book, Section 1.1. "Thermal archaeology" example)
- Similarly, estimating distributions of parameters affecting the diffusion, based on the outcome of diffusion is difficult. (Ill-posed inverse problems in electrical impedance tomography, optical tomography, thermal diffusion tomography, etc.)
- Closely related: image deblurring problems (another classical inverse problem)

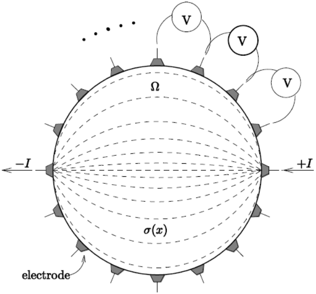
# Electrical impedance tomography (EIT)

- In EIT electric currents *I* are applied to electrodes on the surface of the object and the resulting potentials *V* are measured using the same electrodes.




- The conductivity distribution $\sigma = \sigma(x)$ is reconstructed based on the potential measurements.
- Diffusive tomography modality

# Forward model for EIT

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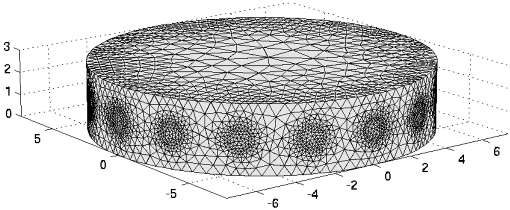
\begin{aligned}
\nabla \cdot (\sigma \nabla u) &= 0\ ,\ x \in \Omega \\
u + z_\ell \sigma \frac{\partial u}{\partial n} &= U_\ell,\ x \in e_\ell \\
\int_{e_\ell} \sigma \frac{\partial u}{\partial n}\ dS &= I_\ell\ ,\ \ell = 1, 2, ..., L \\
\sigma \frac{\partial u}{\partial n} &= 0\ ,\ x \in \partial\Omega \backslash \bigcup_{\ell=1}^{L} e_\ell
\end{aligned}
$$

# Forward model & inverse problem in EIT

- Finite element (FE) approximation of the complete electrode model $\Rightarrow V = U(\sigma)$




- Additive noise model

$$V_{\text{obs}} = U(\sigma) + n$$

- Solving $\sigma$ based on noisy observations $V$ is a non-linear ill-posed inverse problem.

# Example: EIT imaging of concrete

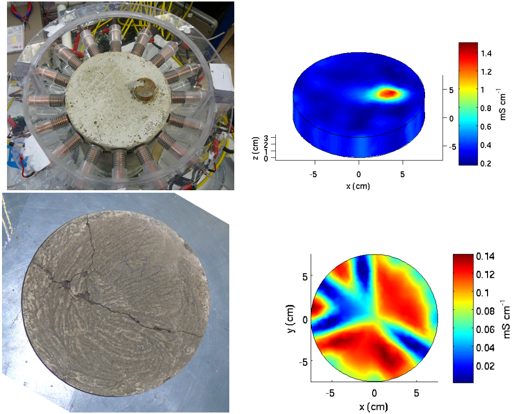



**Figure: EIT imaging of concrete (Karhunen et al 2010)**

# Forward model & inverse problem in EIT

- EIT is a classical inverse problem; has been widely studied theoretically.
- Lot of applications:
  - Process industry (multi-phase flows, single-phase flows)
  - Geophysis (explosives, archeology, soil water content, ect)
  - Medical imaging (breast cancer, lungs, brains)
  - Concrete (cracks, reinforcing bars, humidity, etc)
  - etc
- In this course, many of the examples on large scale inverse problems are related to EIT. Note however, that many of the methods are directly applicable to other large scale inverse problems.