Statistical and Computational Inverse Problems with Applications

Part 5B: Electrical impedance tomography

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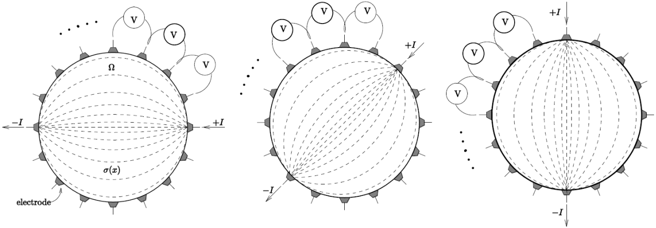
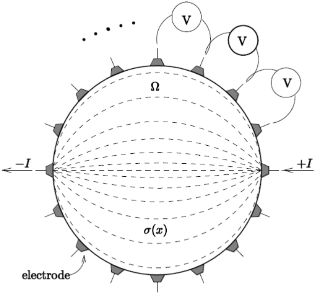
Electrical impedance tomography (EIT)

- In EIT electric currents I are applied to electrodes on the surface of the object and the resulting potentials V are measured using the same electrodes.



- The conductivity distribution $\sigma = \sigma(x)$ is reconstructed based on the potential measurements.
- Diffusive tomography modality

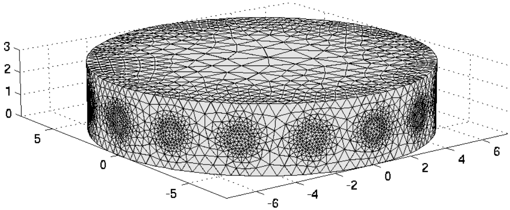
Forward model for EIT



$$
\begin{aligned}
\nabla \cdot (\sigma \nabla u) &= 0\ ,\ x \in \Omega \\
u + z_\ell \sigma \frac{\partial u}{\partial n} &= U_\ell,\ x \in e_\ell \\
\int_{e_\ell} \sigma \frac{\partial u}{\partial n}\ dS &= I_\ell\ ,\ \ell = 1, 2, ..., L \\
\sigma \frac{\partial u}{\partial n} &= 0\ ,\ x \in \partial\Omega \setminus \bigcup_{\ell=1}^{L} e_\ell
\end{aligned}
$$

Forward model & inverse problem in EIT

- Finite element (FE) approximation of the complete electrode model $\Rightarrow V = U(\sigma)$

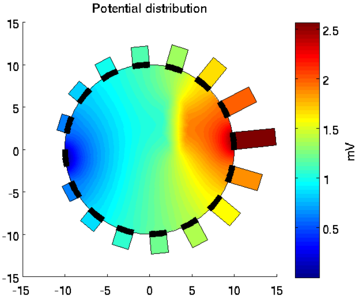


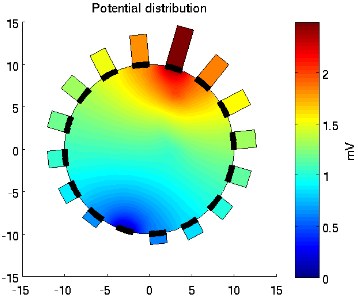
- Additive noise model

$$V_{\text{obs}} = U(\sigma) + n$$

Examples of forward solutions

- See examples of EIT forward solutions in **Appendix 1.** (Don't print it out; huge number of pages & figs.)





- What do the last examples tell us about the ill-posedness of EIT?
- Any suggestions for the remedy?

MAP estimates

- In the case of **Gaussian likelihood model** and **Gibbs' type prior**, the posterior density is of the form

$$
\begin{aligned}
\pi(\sigma|V) &\propto \pi(V|\sigma)\pi(\sigma) \\
&\propto \exp\left(-\frac{1}{2}(V - U(\sigma))^{\mathrm{T}}\Gamma_n^{-1}(V - U(\sigma)) - \frac{1}{2}G(\sigma)\right)
\end{aligned}
$$

- And the MAP estimate can be written in the form

$$
\sigma_{\mathrm{MAP}} = \arg\min_{\sigma}\{\|L_n(V - U(\sigma))\|^2 + G(\sigma)\} \qquad (1)
$$

where $L_n^T L_n = \Gamma_n^{-1}$.

- Iterative solution (e.g. Gauss-Newton)

MAP estimates with Gaussian models

- In the case of **Gaussian likelihood model** and **Gaussian prior**, the posterior density is of the form

$$\pi(\sigma|V) \propto \exp\left(-\frac{1}{2}(V - U(\sigma))^{\mathrm{T}}\Gamma_n^{-1}(V - U(\sigma)) - \frac{1}{2}(\sigma - \eta_\sigma)^{\mathrm{T}}\Gamma_\sigma^{-1}(\sigma - \eta_\sigma)\right)$$

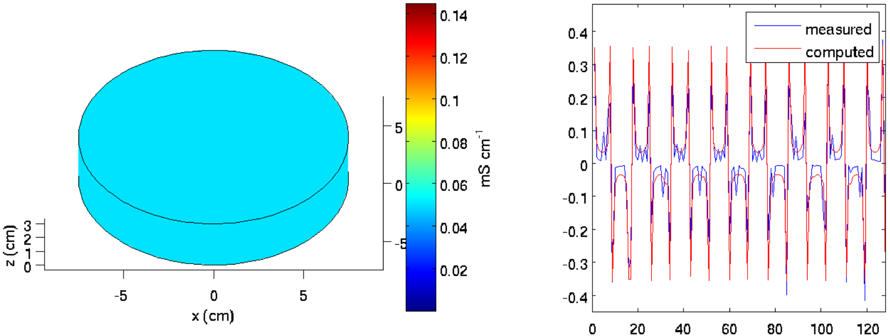
- And the MAP estimate can be written in the form

$$\sigma_{\mathrm{MAP}} = \arg\min_\sigma\{\|L_n(V - U(\sigma))\|^2 + \|L_\sigma(\sigma - \eta_\sigma)\|^2\} \quad (2)$$

where $L_n^T L_n = \Gamma_n^{-1}, \;\; L_\sigma^T L_\sigma = \Gamma_\sigma^{-1}$.

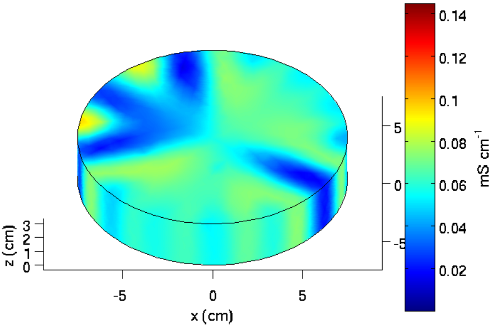
- Iterative solution (e.g. Gauss-Newton)

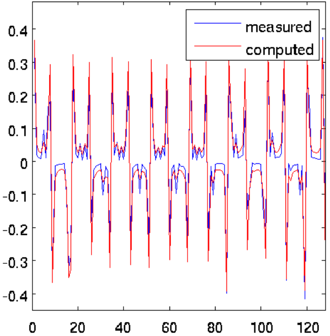
Iteration step 1



Left: estimated conductivity distribution. Right: Measured vs. computed potentials.

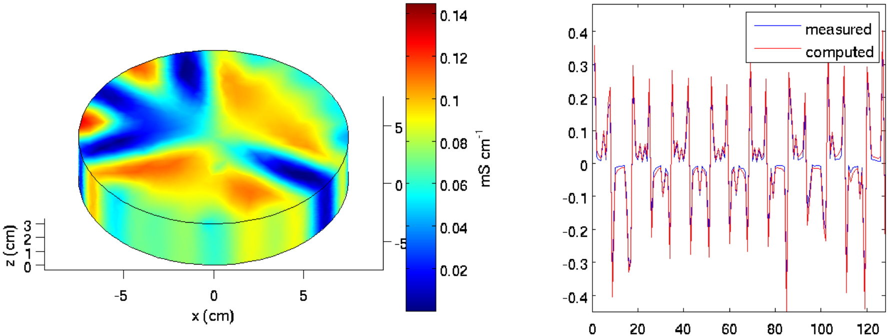
Iteration step 2





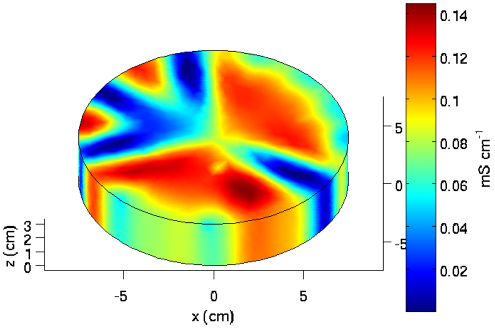
Left: estimated conductivity distribution. Right: Measured vs. computed potentials.

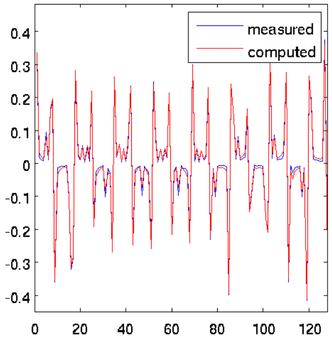
Iteration step 3



Left: estimated conductivity distribution. Right: Measured vs. computed potentials.

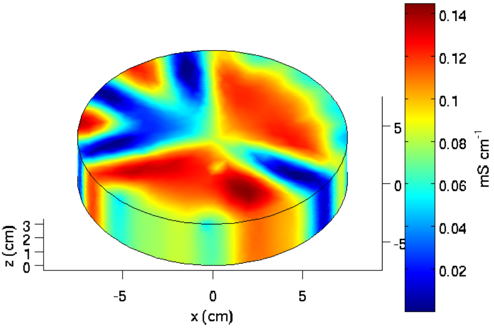
Iteration step 4

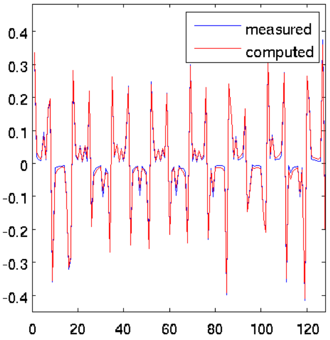




Left: estimated conductivity distribution. Right: Measured vs. computed potentials.

Iteration step 5





Left: estimated conductivity distribution. Right: Measured vs. computed potentials.

MAP estimate

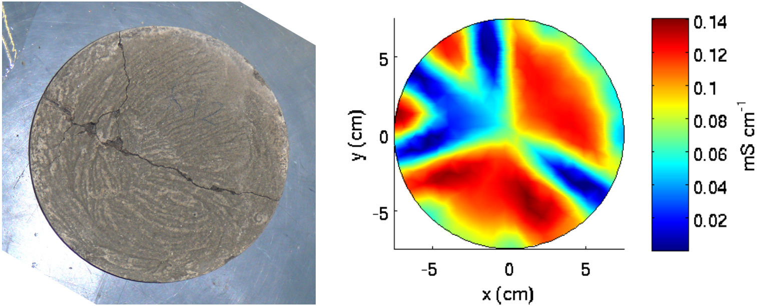


Figure: Left: Photo of the true target; Right: estimated conductivity distribution.

Computational aspects

- Solution of the optimization problem in the MAP estimate typically Gauss-Newton-type iteration
- Line-search
- Non-negativity constraint:
  - e.g. for Gaussian priors, $\mathrm{P}(\sigma < 0) \neq 0$. However, in reality the conductivity is non-negative.
  - In MAP estimates, the non-negativity constraint can be handled by *constrained optimization*

$$\sigma_{\mathrm{MAP}} = \arg\min_{\sigma \geq 0} \{ \| L_n(V - U(\sigma)) \|^2 + \| L_\sigma(\sigma - \eta_\sigma) \|^2 \} \qquad (3)$$

  - Projected line-search (not a good choice...)
  - Interior point method

Interior point method for the non-negativity constraint

- Idea: set a barrier function $b(\sigma)$ which gives high penalty, when any element of the conductivity vector $\sigma_k \to 0$.
- The MAP estimate with the interior point method

$$\sigma_{\mathrm{MAP}} = \arg\min_{\sigma}\{\|L_n(V - U(\sigma))\|^2 + \|L_\sigma(\sigma - \eta_\sigma)\|^2 + b(\sigma)\}$$

- Example: logarithmic barrier function

$$b(\sigma) = -\mu \sum_k^N \ln(\sigma_k) \tag{4}$$

  where μ is a weighting parameter.
- Usually μ is adaptively decreased during the iteration.

White noise prior

- A white noise prior is of the form

$$\sigma \sim \mathcal{N}(\eta_\sigma, \gamma_\sigma^2 I) \tag{5}$$

where η_σ and γ_σ^2 are the expectation and variance of σ.

- The prior density

$$\begin{aligned}\pi(\sigma) &\propto \exp\left(-\frac{1}{2\gamma_\sigma^2}(\sigma - \eta_\sigma)^{\mathrm{T}}(\sigma - \eta_\sigma)\right) &(6)\\ &\propto \exp\left(-\frac{1}{2\gamma_\sigma^2}\|(\sigma - \eta_\sigma)\|^2\right) &(7)\end{aligned}$$

White noise prior: How to select η_σ and γ_σ^2?

- Gaussian random variable σ

$$\sigma \sim \mathcal{N}(\eta_\sigma, \gamma_\sigma^2 I)$$

- Define $\sigma_{\min} = \eta_\sigma - 3\gamma_\sigma$ and $\sigma_{\max} = \eta_\sigma + 3\gamma_\sigma$.
- Then, $\mathrm{P}(\sigma_{\min} < \sigma < \sigma_{\max}) \approx 0.997$.
- Practical way of selecting η_σ and γ_σ^2:
    - The expectation of the conductivity η_σ can often be assessed based on the knowledge of the physical properties of the target (prior information!)
    - Further, you may also have an idea of "upper limit" of conductivity $\sigma_{\max}$ (loosely speaking!)
    - Then, a reasonable choice for the variance is $\gamma_\sigma^2 = \left(\frac{\sigma_{\max} - \eta_\sigma}{3}\right)^2$
- Problems: White noise prior is usually not a good model in EIT – the conductivity is usually spatially correlated.

Uninformative smoothness prior

- Standard (uninformative) smoothness prior (continuous σ)

$$\pi(\sigma) \propto \exp\left(-\alpha \int_\Omega \|\nabla\sigma\|^2 \mathrm{d}r\right)$$

- Finite dimensional approximation for σ; prior density can be written in the form

$$\pi(\sigma) \propto \exp\left(-\frac{1}{2}\alpha\|L_\sigma\sigma\|^2\right) = \exp\left(-\frac{1}{2}\alpha\|\sigma^{\mathrm{T}}L_\sigma^{\mathrm{T}}L_\sigma\sigma\|^2\right)$$

  Matrix $L_\sigma^{\mathrm{T}}L_\sigma$ is not invertible $\Rightarrow$ Γ_σ does not exist.

- Problems: How to select α? How to control the degree of spatial smoothness?

Extensions of the uninformative smoothness prior

- (Uninformative) anisotropic smoothness prior is defined accordingly (continuos form)

$$\pi(\sigma) \propto \exp\left(-\alpha \int_{\Omega} \|A(r)\nabla\sigma\|^2 \mathrm{d}r\right) \tag{8}$$

  where $A(r)$ is tensor field.
- (Uninformative) structural priors can be constructed by selection of $A(r)$ based on structural information (example: anatomical information provided by another imaging modality).

An informative smoothness prior

- Gaussian random variable $\sigma \sim \mathcal{N}(\eta_\sigma, \Gamma_\sigma)$

$$\pi(\sigma) \propto \exp\left(-\frac{1}{2}(\sigma - \eta_\sigma)^{\mathrm{T}}\Gamma_\sigma^{-1}(\sigma - \eta_\sigma)\right) \tag{9}$$

- Write the covariance matrix Γ_σ as

$$\Gamma_\sigma(i,j) = a \exp\left\{-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|_2^2}{2b^2}\right\} \tag{10}$$

  where $\mathbf{x}_i \in \mathbb{R}^{2,3}$ is the spatial coordinate corresponding to a discrete conductivity value σ_i (*Lieberman, Willcox, Ghattas 2010*).

- Other similar models exist.

An informative smoothness prior: How to select *a* and *b*?

- The variance of the conductivity at point $\mathbf{x}_i$ is

$$\mathrm{var}(\gamma_i) = \Gamma_\sigma(i,i) = a \tag{11}$$

- Selection of the variance: See the white noise prior above.
- Define the *correlation length* ℓ as the distance where the cross-covariance $\Gamma_\sigma(i,j)$ drops to 1% of $\mathrm{var}(\gamma_i)$. Then

$$b = \frac{\ell}{\sqrt{2\ln(100)}}. \tag{12}$$

An informative anisotropic smoothness prior

- Again, Gaussian random variable $\sigma \sim \mathcal{N}(\eta_\sigma, \Gamma_\sigma)$

$$\pi(\sigma) \propto \exp\left(-\frac{1}{2}(\sigma - \eta_\sigma)^{\mathrm{T}} \Gamma_\sigma^{-1} (\sigma - \eta_\sigma)\right) \qquad (13)$$

- Write the covariance matrix Γ_σ as

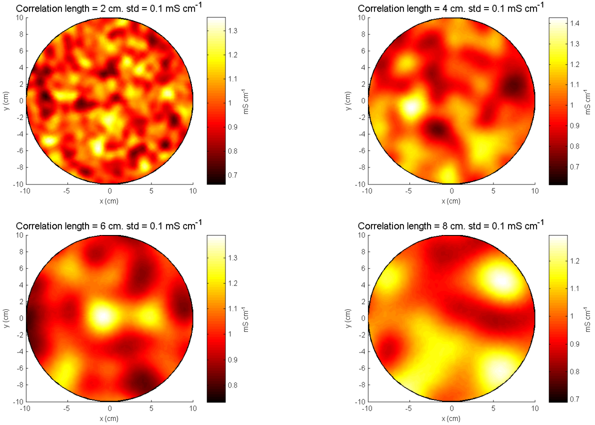
$$\Gamma_\sigma(i,j) = a \exp\left\{ -\frac{\sum_{k=1}^{3} \|\mathbf{x}_i^{(k)} - \mathbf{x}_j^{(k)}\|_2^2}{2b_k^2} \right\} \qquad (14)$$

  where $\mathbf{x}_i \in \mathbb{R}^3, \ \mathbf{x}_i = (\mathbf{x}_i^{(1)}, \mathbf{x}_i^{(2)}, \mathbf{x}_i^{(3)})$ is the spatial coordinate corresponding to a discrete conductivity value σ_i, and coefficients b_k define the correlation lengths ℓ_k at the directions of the coordinate axes.

- Other directions by coordinate transformations.

Examples of informative smoothness priors

- For examples of informative smoothness priors, see **Appendix 2.**



Samples corresponding to smoothness priors with different correlation lenghts.

A sample based Gaussian prior

- Assume that you have a set of samples of the conductivity distribution (based on e.g. other experiments or a flow simulation); denote the samples by $\sigma^{(j)},\ j = 1, \ldots, K$.
- Approximate σ as a Gaussian random variable $\sigma \sim \mathcal{N}(\eta_\sigma, \Gamma_\sigma)$

$$\pi(\sigma) \propto \exp\left(-\frac{1}{2}(\sigma - \eta_\sigma)^{\mathrm{T}}\Gamma_\sigma^{-1}(\sigma - \eta_\sigma)\right) \tag{15}$$

where η_σ is chosen to be the sample mean $\frac{1}{K}\sum_{j=1}^{K}\sigma^{(j)}$, and the sample covariance is used as the prior covariance matrix:

$$\Gamma_\sigma = \frac{1}{K-1}\sum_{i=1}^{K}(\sigma^{(i)} - \eta_\sigma)(\sigma^{(i)} - \eta_\sigma)^{\mathrm{T}} \tag{16}$$

Total variation prior

- A couple of different versions of TV prior exists. The following one has certain advantages.
- Total variation prior (continuous form, 2D case)

$$\pi(\sigma) \propto \exp\left(-\alpha \int_\Omega \|\nabla\sigma\|_2 \mathrm{d}r\right) = \exp\left(-\alpha \int_\Omega \sqrt{\frac{\partial\sigma}{\partial x} + \frac{\partial\sigma}{\partial y}} \mathrm{d}r\right)$$

- Finite dimensional approximation

$$\pi(\sigma) \propto \exp\left(-\alpha \sum_{\ell=1}^{M} \sqrt{(L_x\sigma)_\ell^2 + (L_y\sigma)_\ell^2}\right)$$

- Promotes sparsity of $\nabla\sigma$.

Total variation prior

- Hence

$$\pi(\sigma) \propto \exp\left(-\alpha A(\sigma)\right)$$

where

$$A(\sigma) = \sum_{\ell=1}^{M} \sqrt{(L_x\sigma)_\ell^2 + (L_y\sigma)_\ell^2}$$

- Gibbs' type prior
- The posterior density is of the form

$$\begin{aligned} \pi(\sigma|V) &\propto \pi(V|\sigma)\pi(\sigma) \\ &\propto \exp\left(-\frac{1}{2}(V - U(\sigma))^{\mathrm{T}}\Gamma_n^{-1}(V - U(\sigma)) - A(\sigma)\right) \end{aligned}$$

Total variation prior

- MAP estimate

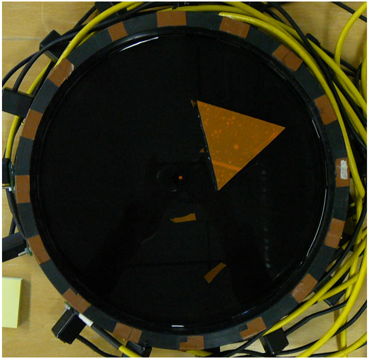
$$\sigma_{\mathrm{MAP}} = \arg\min_{\sigma}\{\frac{1}{2}\|L_n(V - U(\sigma))\|^2 + A(\sigma)\}$$

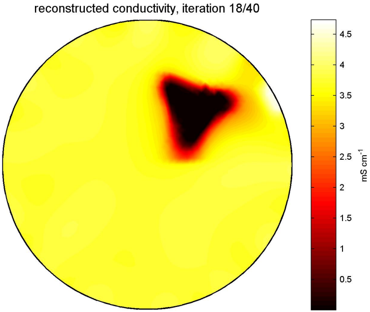
- Solution: e.g. Gauss-Newton
- Note: $A(\sigma)$ is not differentiable. Hence, approximation:

$$A(\sigma) = \sum_{\ell=1}^{M} \sqrt{(L_x\sigma)_\ell^2 + (L_y\sigma)_\ell^2 + \beta}$$

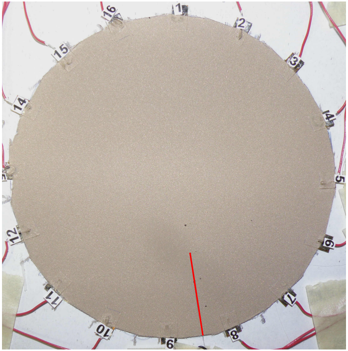
where β is a small constant.

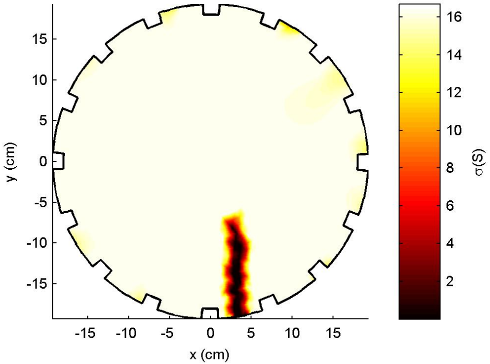
An example




reconstructed conductivity, iteration 18/40
4.5
4
3.5
3
2.5
2
1.5
1
0.5
mS cm^{-1}

Sensing skin application





- http://iopscience.iop.org/0964-1726/23/8/085001/article
- http://phys.org/news/2014-06-skin-quickly-concrete.html

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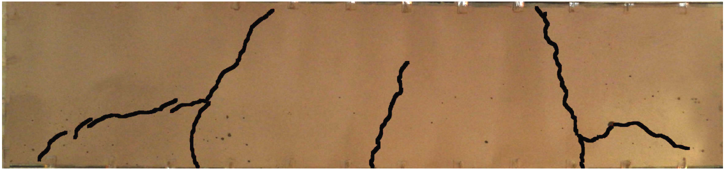

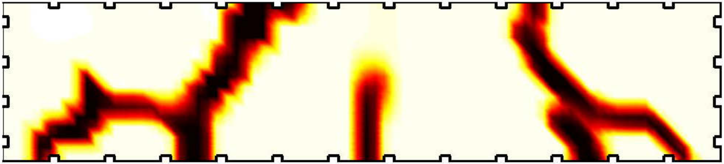

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