STOKASTIIKAN PERUSTEET Harjoitus 1

1.1 (a) $\mathbb{P}(\emptyset) = 0$:

$\Omega \cap \emptyset = \emptyset$, joten $\mathbb{P}(\Omega) = \mathbb{P}(\Omega \cup \emptyset) \overset{1.8\,(ii)}{=} \mathbb{P}(\Omega) + \mathbb{P}(\emptyset)$.

Siten $\mathbb{P}(\emptyset) = \mathbb{P}(\Omega) - \mathbb{P}(\Omega) = 0$

(b) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$:

$A^c \cup A = \Omega$ ja $A^c \cap A = \emptyset$, joten

$1 = \mathbb{P}(\Omega) = \mathbb{P}(A^c \cup A) = \mathbb{P}(A^c) + \mathbb{P}(A)$ ja edelleen

$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$)

(c) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$:

$B = (A \cap B) \cup (A^c \cap B)$, $(A \cap B) \cap (A^c \cap B) = \emptyset$

$A \cup B = A \cup (A^c \cap B)$, $A \cap (A^c \cap B) = \emptyset$

$$\Rightarrow \begin{cases} \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(A^c \cap B) \\ \mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) \end{cases}$$

$$\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

(d) $0 \leq \mathbb{P}(A) \leq 1$

$\mathbb{P}(A) \geq 0$ aina

$0 \leq \mathbb{P}(A^c) = 1 - \mathbb{P}(A) \Rightarrow \mathbb{P}(A) \leq 1$

(e) $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$:

$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) \overset{A \subset B}{=} \mathbb{P}(A) + \underbrace{\mathbb{P}(A^c \cap B)}_{\geq 0} \geq \mathbb{P}(A)$

(f) $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$B = (B \setminus A) \cup (A \cap B)$, $(B \setminus A) \cap (A \cap B) = \emptyset$

$\Rightarrow \mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B) \Rightarrow \mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$

1.2 (a) $\Omega = \{1, 2, 3, \dots\}$ $p \in (0,1]$, $P(\omega) = (1-p)^{\omega-1} p$

(i) $P(\omega) = \underbrace{(1-p)^{\omega-1}}_{\geq 0} \underbrace{p}_{\geq 0} \geq 0$

(ii) $\sum_{\omega=1}^{\infty} \mathbb{P}(\omega) = \sum_{\omega=1}^{\infty} (1-p)^{\omega-1} p = p \sum_{k=0}^{\infty} (1-p)^k \overset{\text{geom summa}}{=} p \frac{1}{1-(1-p)} = \frac{p}{p} = 1$

(b) $\Omega = \{0, 1, 2, \dots\}$, $\lambda > 0$, $P(\omega) = e^{-\lambda} \frac{\lambda^\omega}{\omega!}$

(i) $P(\omega) = \underbrace{e^{-\lambda}}_{\geq 0} \underbrace{\frac{\lambda^\omega}{\omega!}}_{\geq 0} \geq 0$

(ii) $\sum_{\omega=0}^{\infty} e^{-\lambda} \frac{\lambda^\omega}{\omega!} = e^{-\lambda} \sum_{\omega=0}^{\infty} \frac{\lambda^\omega}{\omega!} = e^{-\lambda} e^{\lambda} = 1$

(c) $\Omega = \{\omega_1, \dots, \omega_N\}$, $N \in \mathbb{N}$, $P(\omega) = \frac{1}{|\Omega|} \geq 0$

$\sum_{i=1}^{N} P(\omega) = \sum_{i=1}^{N} \frac{1}{|\Omega|} = N \cdot \frac{1}{N} = 1$

GEOM sarjan $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ $\quad a = p$, $r = 1-p$

$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \Rightarrow e^\lambda = \sum \frac{\lambda e^0}{k!} \lambda^k = \sum \frac{\lambda^k}{k!}$

1.3. Olk μ_i tasajakauma joukossa S_i, $i \in \{1,2\}$, eli

$$\mu(s_i) = \frac{1}{|S_i|} \quad \text{kaikilla} \quad s_i \in S_i \quad i \in \{1,2\}.$$

Jakauma $\mu = \mu_1 \times \mu_2$ on määritelty joukossa $S = S_1 \times S_2$, jossa on $|S| = |S_1||S_2|$ alkiota, ja se on määritelty kaavalla

$$\mu(s) = (\mu_1 \times \mu_2)(s) = \mu_1(s_1)\mu_2(s_2) = \frac{1}{|S_1|}\frac{1}{|S_2|} = \frac{1}{|S|}.$$

Siten μ on tasajakauma joukossa S.

Numeroituvasti äärettömässä joukossa ei voi määritellä tasajakaumaa sillä äärettömän monen yhtä suuren todennäköisyyden summa voi olla ainoastaan 0 tai ∞, mutta ei 1

1.4 Olk. $A_1, \dots, A_k \in \Omega = \{\omega_1, \dots, \omega_n\}$

(a) Tarkastellaan joukkoja $B_1, \dots, B_k$, missä $B_i = A_j \setminus \bigcup_{j \neq k} A_i \subset A_i$. Silloin $B_i \cap B_j = \emptyset$ kaikilla $i \neq j$ ja $\bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A_i$.

Nyt $$\mathbb{P}\left(\bigcup_{i=1}^{k} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{k} B_i\right) = \sum_{i=1}^{k} \overbrace{\mathbb{P}(B_i)}^{\substack{B_i \subset A_i \\ \leq \mathbb{P}(A_i)}} \leq \sum_{i=1}^{k} \mathbb{P}(A_i) \quad \square$$

(b) Olk $\Omega = \{1, \dots, 100\}$, $A_1 = \{1,2\}$, $A_2 = \{5\}$. ja $\mathbb{P}(\{\omega\}) = \frac{1}{100}$ $\forall \omega$

$$\mathbb{P}\left(\bigcup_{i=1}^{2} A_i\right) = \frac{3}{100} = \mathbb{P}(A_1) + \mathbb{P}(A_2)$$

(c) olk. $A_1 = \{1,2\}$, $A_2 = \{2,3\}$

$$\mathbb{P}\left(\bigcup_{i=1}^{2} A_i\right) = \frac{3}{100} \neq \frac{4}{100} = \mathbb{P}(A_1) + \mathbb{P}(A_2)$$

1.5 $\mathbb{P}(A) > 0$, $\mathbb{P}(B) > 0$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} \frac{\cancel{\mathbb{P}(B)}}{\mathbb{P}(A)} \frac{\mathbb{P}(A \cap B)}{\cancel{\mathbb{P}(B)}}$$

$$= \frac{\mathbb{P}(A)}{\mathbb{P}(B)} \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)\mathbb{P}(B \mid A)}{\mathbb{P}(B)}$$

1.6 $S_1 = \{-150, 0, 75\}$, $S_2 = \{-200, 0, 100\}$

$\mu_1(s_1) = \frac{1}{3}$ $s_1 \in S_1$, $\mu_2(s_2) = \frac{1}{3}$ $s_2 \in S_2$

$S = S_1 \times S_2$, $\mu = \mu_1 \times \mu_2 \Rightarrow \mu(s) = \frac{1}{9}$ $\forall s \in S$

Jos lainojen takaisinmaksut eivät riipu toisistaan, niin tuottojakauma P on tasajakauma joukossa
$T = \{-350, -200, -150, -125, -50, 0, 75, 100, 175\}$
Voitolle jäämisen todennäköisyys on siis $\frac{3}{9} = \frac{1}{3}$ ja tappion tn. on $\frac{5}{9}$.