STOKASTIIKAN PERUSTEET Harjoitus 2

2.1. (a) Olkoon $\Omega = \{\text{ei läpäise, läpäisee}\} = \{\omega_1, \omega_2\}$, $P(\omega) = \frac{1}{2}$, $\forall \omega \in \Omega$,
ja $X_i : \Omega \to \{0,1\}$, $X(\omega) = \begin{cases} 0 & \omega = \omega_1 \\ 1 & \omega = \omega_2 \end{cases}$

$$\mathbb{P}(\text{kaikki 5 läpäisee}) = \mathbb{P}(X_1 \geq 1, X_2 \geq 1, \dots, X_5 \geq 1) \overset{''}{=} \mathbb{P}(\min\{X_i, i=1,\dots,n\} \geq 1)$$
$$\overset{\perp\!\!\!\perp}{=} \prod_{i=1}^{5} \mathbb{P}(X_i \geq 1) = \left(\tfrac{1}{2}\right)^5 = \frac{1}{2^5} = \frac{1}{32} = 0.03125$$

(b) $$\mathbb{P}(\min\{X_i, i=1,\dots,n\} \geq x) = \mathbb{P}(X_i \geq x \ \forall i=1,\dots,n)$$
$$= \mathbb{P}(X_1 \geq x, X_2 \geq x, \dots, X_n \geq x)$$
$$\overset{\perp\!\!\!\perp}{=} \prod_{i=1}^{n} \mathbb{P}(X_i \geq x)$$

2.2. $\mathbb{P}(\Theta_i = 1) = p$, $\mathbb{P}(\Theta_i = 0) = 1-p$, $p \in (0,1)$

(a) $X = \Theta_1 + \Theta_2 \in \{0,1,2\}$
$$\mathbb{P}(X=0) = \mathbb{P}(\Theta_1 = 0, \Theta_2 = 0) = \mathbb{P}(\Theta_1 = 0)\mathbb{P}(\Theta_2 = 0) = (1-p)^2$$
$$\mathbb{P}(X=1) = \mathbb{P}(\{\Theta_1 = 0, \Theta_2 = 1\} \cup \{\Theta_1 = 1, \Theta_2 = 0\}) = \mathbb{P}(\Theta_1 = 0, \Theta_2 = 1) + \mathbb{P}(\Theta_1 = 1, \Theta_2 = 0)$$
$$= \mathbb{P}(\Theta_1 = 0)\mathbb{P}(\Theta_2 = 1) + \mathbb{P}(\Theta_1 = 1)\mathbb{P}(\Theta_2 = 0) = 2p(1-p) = \binom{2}{1}p^1(1-p)^{2-1}$$
$$\mathbb{P}(X=2) = \mathbb{P}(\Theta_1 = 1, \Theta_2 = 1) = \mathbb{P}(\Theta_1 = 1)\mathbb{P}(\Theta_2 = 1) = p^2$$
$$\therefore \ \mathbb{P}(X=k) = \binom{k}{2}p^k(1-p)^{2-k} \quad k \in \{0,1,2\}$$
Binomijakauma $Bin(2,p)$

(b) $Y = \Theta_1\Theta_2 \in \{0,1\}$
$$\mathbb{P}(Y=0) = \mathbb{P}(\Theta_1 = 0 \text{ tai } \Theta_2 = 0) = \mathbb{P}(\{\Theta_1 = 0\} \cup \{\Theta_1 = 1, \Theta_2 = 0\})$$
$$= \mathbb{P}(\Theta_1 = 0) + \mathbb{P}(\Theta_1 = 1)\mathbb{P}(\Theta_2 = 0) = 1-p + p(1-p) = 1-p^2$$
$$\mathbb{P}(Y=1) = \mathbb{P}(\Theta_1 = 1 \text{ ja } \Theta_2 = 1) = \mathbb{P}(\Theta_1 = 1, \Theta_2 = 1) = \mathbb{P}(\Theta_1 = 1)\mathbb{P}(\Theta_2 = 1) = p^2$$
$$\Rightarrow \ Y \sim \text{BERNOULLI}(p^2)$$

(c) $Z = \Theta_1 + \dots + \Theta_n$
Arvataan (a)-kohdan perusteella, että $\mathbb{P}(Z=k) = \binom{n}{k}p^k(1-p)^{n-k}$
Todistetaan kaava induktiolla:
(1) Väite pätee, jos $n=1$ (tai $n=2$).
(2) Oletetaan, että väite pätee $n-1$:lle kaikilla $k \in \{0,\dots,n-1\}$, $n \geq 2$
ok $k \geq 1$
$$\mathbb{P}(Z=k) = \mathbb{P}(\Theta_1 + \dots + \Theta_n = k) = \mathbb{P}(\Theta_1 + \dots + \Theta_{n-1} = k, \Theta_n = 0)$$
$$+ \mathbb{P}(\Theta_1 + \dots + \Theta_{n-1} = k-1, \Theta_n = 1)$$
$$\overset{\perp\!\!\!\perp}{=} \mathbb{P}(\Theta_1 + \dots + \Theta_{n-1} = k)\mathbb{P}(\Theta_n = 0) +$$
$$\mathbb{P}(\Theta_1 + \dots + \Theta_{n-1} = k-1)\mathbb{P}(\Theta_n = 1)$$
$$= \binom{n-1}{k}p^k(1-p)^{n-1-k}(1-p) + \binom{n-1}{k-1}p^{k-1}(1-p)^{n-1-(k-1)}p$$
$$= \left(\binom{n-1}{k} + \binom{n-1}{k-1}\right)p^k(1-p)^{n-k} \overset{\text{VIHJE}}{=} \binom{n}{k}p^k(1-p)^{n-k}$$
Lisäksi
$$\mathbb{P}(Z=0) = \mathbb{P}(\Theta_1 + \dots + \Theta_n = 0) = \prod_{i=1}^{n}\mathbb{P}(\Theta_i = 0) = (1-p)^n = \binom{n}{0}p^0(1-p)^{n-0}$$
$$\therefore \ Z \sim Bin(n,p)$$

$\overset{2}{\to}$ (d) $W = \theta_1 \cdots \theta_n \in \{0,1\}$

$\mathbb{P}(W=1) = \mathbb{P}(\theta_i = 1 \text{ kaikilla } i=1,\dots,n) \overset{\perp\!\!\!\perp}{=} \prod_{i=1}^{n} \mathbb{P}(\theta_i = 1) = p^n$

$\mathbb{P}(W=0) = \mathbb{P}(\theta_i = 0 \text{ jollain } i=1\dots n) = 1 - \mathbb{P}(\theta_i = 1 \text{ kaikilla } i=1,\dots,n)$

$= 1 - p^n$

$\therefore\ W \sim \text{BERNOULLI}(p^n)$ $\square$

3. (a) $\mathbb{P}(X_i \in A_i, X_j \in A_j, i \neq j) = \mathbb{P}(X_i \in A_i, X_j \in A_j, X_k \in \Omega,\ i \neq j, k \notin \{i,j\})$

$\overset{\perp\!\!\!\perp}{=} \mathbb{P}(X_i \in A_i)\mathbb{P}(X_j \in A_j)\underbrace{\mathbb{P}(X_k \in \mathbb{Z})}_{1}$

$= \mathbb{P}(X_i \in A_i)\mathbb{P}(X_j \in A_j)$

$\therefore$ jos X_1, X_2, X_3 ovat riippumattomat, niin X_i, X_j ovat $\perp\!\!\!\perp$ kaikilla $i \neq j$

(b) Oletetaan, että $(X_1, X_2, X_3) \in \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (1,1,1), (2,2,2), (3,3,3)\}$

Sillon $\mathbb{P}(X_i = k) = \frac{1}{3}$ kaikilla $i, k \in \{1,2,3\}$ ja

$\mathbb{P}(X_i = k, X_j = \ell) = \frac{1}{9} = \mathbb{P}(X_i = k)\mathbb{P}(X_j = \ell) \quad \forall\ i,j,k,\ell \in \{1,2,3\}, i \neq j$

mutta

$\mathbb{P}(X_1 = 1, X_2 = 2, X_3 = 2) = 0 \neq \frac{1}{27} = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 2)\mathbb{P}(X_3 = 3).$

Eli parien riippumattomuuksista ei seuraa kolmikon riippumattomuutta

4. $\mathbb{P}(X = s) > 0 \quad \forall\ s.$

<u>Vä</u> $\mathbb{P}(Y = t \mid X = s) = \mathbb{P}(Y = t) \iff \mathbb{P}(Y = t, X = s) = \mathbb{P}(Y = t)\mathbb{P}(X = s).$

Merk $A = \{Y = t\}$, $B = \{X = s\}$ $\mathbb{P}(B) > 0$

"$\Rightarrow$" $\mathbb{P}(A) = \mathbb{P}(A \mid B) \overset{\text{MÄÄR.}}{=} \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$

"$\Leftarrow$" $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \dfrac{\mathbb{P}(A, B)}{\mathbb{P}(B)} = \dfrac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A)$ $\square$

5. (a) Olk. $A = \{X + Y = 175\}$, $B = \{X = 0\}$. Sillon

$\mathbb{P}(A) = \frac{1}{9}$, mutta $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(\overbrace{\{X + Y = 175, X = 0\}}^{=\emptyset})}{\mathbb{P}(X = 0)} = 0$

(b) Olk. $A = \{Y = 0\}$; $B = \{X = 0\}$, $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{3}$

$\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A, B)}{\mathbb{P}(B)} \overset{X \perp\!\!\!\perp Y}{=} \dfrac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A)$

(c) Olk $A = \{X + Y = 0\}$ $B = \{X = 0\}$

$\mathbb{P}(A) = \frac{1}{9}$, mutta

$\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A, B)}{\mathbb{P}(B)} = \dfrac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$