Stokastiikan perusteet H4

1 Olkoot X ja Y satunnaismuuttujat, jotka kuvaavat Pekan ja Antin maksamia summia. X on tasajakautunut joukossa $S_1 = \{-150, 0, 75\}$ ja Y joukossa $S_2 = \{-200, 0, 100\}$. $X+Y$ on puolestaan tasajakautunut joukossa $T = \{-350, -200, -150, -125, -50, 0, 75, 100, 175\}$

(a)
$$P_{(X+Y,Y)}(t, s_2) = \begin{cases} \frac{1}{9} & \text{kun } (t-s_2, s_2) \in S_1 \times S_2 \\ 0 & \text{muualla,} \end{cases}$$
joten
$$P_{X+Y|Y}(t|s_2) = \frac{P_{(X+Y,Y)}(t, s_2)}{P_Y(s_2)} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$$
ja
$$\mathbb{E}[X+Y \mid Y=100] = \sum_{t-100 \in S_1} t P_{X+Y|Y}(t|100) = (\overbrace{175+100-50}^{225})\frac{1}{3} = 75$$

(b) Rajoittumalla joukkoon $\{X \geq 0\} \cap \{Y \geq 0\}$ nähdään, että $X+Y \in \{0, 75, 100, 175\}$ ja $X+Y$ noudattaa tässä joukossa tasajakaumaa. Siten
$$\mathbb{E}[X+Y \mid X \geq 0, Y \geq 0] = \frac{1}{4}(75+100+175) = \frac{350}{4} = 87.5$$

(c) H3T5: $\mathbb{E}[X+Y] = -58\frac{1}{3}$ ja siten
$$\operatorname{var}(X+Y) = \sum_{t \in T}(t+58\tfrac{1}{3})^2 \cdot \frac{1}{9} \approx 24305.56$$
$$\operatorname{sd}(X+Y) = \sqrt{\operatorname{var}(X+Y)} \approx 155.90$$

<u>HUOM</u> Tämä on koko populaatiosta laskettu keskihajonta, joka <u>ei ole</u> sama kuin otoskeskihajonta!

2. (a) $\mathbb{P}(\mathbb{1}_A = 1) = \mathbb{P}(A)$ $\Rightarrow$ $\mathbb{1}_A \sim \text{Bernoulli}(\mathbb{P}(A))$
$\mathbb{P}(\mathbb{1}_A = 0) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

(b) $\mathbb{E}[\mathbb{1}_A] = 1 \cdot \mathbb{P}(A) + 0 \cdot \mathbb{P}(A^c) = \mathbb{P}(A)$
$$\operatorname{var}(\mathbb{1}_A) = \mathbb{E}[(\mathbb{1}_A - \mathbb{E}[\mathbb{1}_A])^2] = \mathbb{E}[\underbrace{\mathbb{1}_A^2}_{\mathbb{1}_A}] - 2\mathbb{E}[\mathbb{1}_A]\mathbb{E}[\mathbb{1}_A] + \mathbb{E}[\mathbb{1}_A]^2$$
$$= \mathbb{P}(A) - \mathbb{P}(A)^2 = \mathbb{P}(A)(1-\mathbb{P}(A))$$

(c) $\mathbb{1}_A$ on BERNOULLI-jakautunut parametrilla $p = \mathbb{P}(A)$, joten $Z \sim \text{Bin}(n,p)$ H2T2(c) nojalla.

(d) $\mathbb{E}[Z] = \sum_{i=1}^{n} \mathbb{E}[\mathbb{1}_{A_i}] = np$
$\operatorname{var}[Z] \overset{\perp}{=} \sum_{i=1}^{n} \operatorname{var}(A_i) = n\operatorname{var}(\mathbb{1}_A) = np(1-p)$ $\square$

4.3

$$\mathrm{var}(\bar{X}) = \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}(X_i) = \frac{n\,\mathrm{var}(X)}{n^2} = \frac{\mathrm{var}(X)}{n}$$

$$\mathbb{E}[V] = \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mathbb{E}[X]-(\bar{X}-\mathbb{E}[X]))^2\right]$$

$$= \frac{1}{n-1}\mathbb{E}\left[\sum_{i=1}^{n}(X_i-\mathbb{E}[X])^2 - 2(\bar{X}-\mathbb{E}[X])\underbrace{\sum_{i=1}^{n}(X_i-\mathbb{E}[X])}_{n(\bar{X}-\mathbb{E}[X])} + \sum_{i=1}^{n}(\bar{X}-\mathbb{E}[X])^2\right]$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}\underbrace{\mathbb{E}[(X_i-\mathbb{E}[X])^2]}_{\mathrm{var}(X)} - 2n\underbrace{\mathbb{E}[(\bar{X}-\mathbb{E}[X])^2]}_{\mathrm{var}(\bar{X})} + n\underbrace{\mathbb{E}[(\bar{X}-\mathbb{E}[X])^2]}_{\mathrm{var}(\bar{X})}\right)$$

$$= \frac{1}{n-1}\left(n\,\mathrm{var}(X) - 2n\underbrace{\mathrm{var}(\bar{X})}_{\frac{\mathrm{var}(X)}{n}} + \underbrace{n\,\mathrm{var}(\bar{X})}_{\frac{\mathrm{var}(X)}{n}}\right)$$

$$= \frac{1}{n-1}\left(n\,\mathrm{var}(X) - \mathrm{var}(X)\right) = \mathrm{var}(X).$$

4.4 $X_1, X_2, \ldots$ i.i.d., $\mathbb{E}[X_1] = 1$, $Y_1, Y_2, \ldots$ i.i.d. $\mathbb{E}[Y_1] = 2$

$$S_n = \begin{cases} X_1 + X_2 + \ldots + X_n & \text{jos saadaan kruuna} \\ Y_1 + Y_2 + \ldots + Y_n & \text{jos saadaan klaava} \end{cases}$$

(a) $$\mathbb{E}[S_n] \overset{L4.13}{=} \mathbb{E}[S_n \mid \text{kruuna}]\,\mathbb{P}(\text{kruuna}) + \mathbb{E}[S_n \mid \text{klaava}]\,\mathbb{P}(\text{klaava})$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} X_i\right]\cdot\frac{1}{2} + \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right]\frac{1}{2} = \frac{n}{2} + \frac{2n}{2} = \frac{3n}{2}$$

$$\mathbb{E}\left[\frac{S_n}{n}\right] = \frac{1}{n}\mathbb{E}[S_n] = \frac{1}{n}\cdot\frac{3n}{2} = \frac{3}{2}$$

(b) Olkoon $Z : \Omega \to \{\text{kruuna}, \text{klaava}\}$ kolikonheitto. ja $A = \{\omega \in \Omega : Z(\omega) = \text{kruuna}\}$.

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \frac{3}{2}\right| > \frac{1}{4} \,\middle|\, A\right) = \frac{\mathbb{P}\left(\left|\frac{S_n}{n} - \frac{3}{2}\right| > \frac{1}{4}, A\right)}{\mathbb{P}(A)}$$

$$= \frac{1}{\mathbb{P}(A)}\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{3}{2}\right| > \frac{1}{4}, A\right)$$

$\{|\frac{1}{n}\sum X_i - 1| \leq \frac{1}{4}\} \subset \{|\frac{1}{n}\sum X_i - \frac{3}{2}| > \frac{1}{4}\}$, $X_i \perp\!\!\!\perp A$

$$\geq \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - 1\right| \leq \frac{1}{4}\right)$$

$$= 1 - \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - 1\right| > \frac{1}{4}\right)$$

$$\geq 1 - \frac{\mathrm{var}(X)}{\frac{1}{4^2}\,n} \xrightarrow{n\to\infty} 1$$

ja

$$\mathbb{P}\Big(\overbrace{\left|\frac{S_n}{n} - \frac{3}{2}\right| > \frac{1}{4}}^{C_n :=}\Big) \overset{(*)}{=} \mathbb{P}(C_n \mid A)\mathbb{P}(A) + \mathbb{P}(C_n \mid A^c)\mathbb{P}(A^c)$$

$$\geq \mathbb{P}(C_n \mid A)\mathbb{P}(A) = \frac{1}{2}\underbrace{\mathbb{P}(C_n \mid A)}_{\to 1} \longrightarrow \frac{1}{2}$$

(*) $$\mathbb{P}(C) = \mathbb{P}(C \cap A) + \mathbb{P}(C \cap A^c)$$

$$= \mathbb{P}(C \mid A)\mathbb{P}(A) + \mathbb{P}(C \mid A^c)\mathbb{P}(A^c)$$

(jos $\mathbb{P}(A) \in (0,1)$)

4.4 (c) Tämä ei ole ristiriidassa suurten lukujen lain kanssa, sillä

$S_n = Z_1 + \ldots + Z_n$, missä

$Z_i = 1\!\!1_A X_i + 1\!\!1_{A^c} Y_i$, jolloin

$$\mathbb{E}[Z_1 Z_2] = \mathbb{E}[(1\!\!1_A X_1 + 1\!\!1_{A^c} Y_1)(1\!\!1_A X_2 + 1\!\!1_{A^c} Y_2)]$$

$1\!\!1_A 1\!\!1_A = 1\!\!1_A$

$1\!\!1_A 1\!\!1_{A^c} = 0$

$$= \mathbb{E}[1\!\!1_A X_1 X_2 + 0 X_1 Y_2 + 0 \cdot X_2 Y_1 + 1\!\!1_{A^c} Y_1 Y_2]$$

$$= \mathbb{E}[1\!\!1_A]\mathbb{E}[X_1]\mathbb{E}[X_2] + \mathbb{E}[1\!\!1_{A^c}]\mathbb{E}[Y_1]\mathbb{E}[Y_2] = \tfrac{1}{2}\cdot 1 \cdot 1 + \tfrac{1}{2}\cdot 2 \cdot 2 = \tfrac{5}{2}$$

$$\mathbb{E}[Z_i] = \mathbb{E}[1\!\!1_A X_i + 1\!\!1_{A^c} Y_i] = \mathbb{E}[1\!\!1_A]\mathbb{E}[X_i] + \mathbb{E}[1\!\!1_{A^c}]\mathbb{E}[Y_i] = \tfrac{1}{2}\cdot 1 + \tfrac{1}{2}\cdot 2 = \tfrac{3}{2}$$

ja $\mathbb{E}[Z_1]\mathbb{E}[Z_2] = \frac{9}{4} \neq \frac{5}{2} = \mathbb{E}[Z_1 Z_2] \implies Z_1 \not\perp\!\!\!\perp Z_2$

4.5 (a) Koska $\mathbb{E}[X] < \infty$, $\mathbb{E}[Y] < \infty$, niin MÄÄRITELMÄN 4.2 nojalla $\mathbb{E}[|X|] < \infty$ ja $\mathbb{E}[|Y|] < \infty$. Olkoon

$Z = aX + bY$, jolloin $|Z| \leq |a||X| + |b||Y|$ ja

$\mathbb{E}[|Z|] \leq |a|\mathbb{E}[X] + |b|\mathbb{E}[Y] < \infty$, eli $\sum_{\omega\in\Omega} |Z(\omega)|P(\omega) < \infty$,

koska positiivitermisen sarjan summausjärjestystä voi vaihtaa

$$\mathbb{E}[|Z|] = \sum_{\omega\in\Omega} |Z(\omega)|P(\omega) = \sum_{\omega\in\Omega} |aX(\omega) + bY(\omega)|P(\omega)$$

$$\leq |a|\sum_{\omega\in\Omega}|X(\omega)|P(\omega) + |b|\sum_{\omega\in\Omega}|Y(\omega)|P(\omega) = a\mathbb{E}[|X|] + b\mathbb{E}[|Y|] < \infty$$

Itseisesti suppenevan sarjan summausjärjestystä voi vaihtaa, joten samaan tapaan kuin yllä nähdään, että

$$\mathbb{E}[Z] = \sum_{\omega\in\Omega} (aX(\omega) + bY(\omega))P(\omega) = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

$\square$

(b)

$$\mathrm{var}(aX+b) = \mathbb{E}[(aX + b - \mathbb{E}[aX+b])^2]$$

$$= \mathbb{E}[(aX - a\mathbb{E}[X] + \underbrace{b-b}_{=0})^2]$$

$$= a^2\mathbb{E}[(X - \mathbb{E}[X])^2] = a^2 \mathrm{var}(X)$$

$\square$