STOKASTIIKAN PERUSTEET H5

5.1 $U_1, \dots, U_{300} \sim \text{tas}\{0, 1, \dots, 36\}$ riippumattomia, $V_0 = 300$

(a) Pelitilin arvo t:n pelikierroksen jälkeen: Olk $A = \{1, \dots, 18\}$

$$V_t = V_0 + \sum_{s=1}^{t} (2\mathbb{1}_A(U_s) - 1)$$

$$\Rightarrow f(u) = 2\sum_{s=1}^{t} \mathbb{1}_A(U_s) - t$$

(b) $V_{300} = V_0 + v - l$, missä v on voitettujen kierrosten määrä ja l on hävittyjen kierrosten määrä. Siten $l = 300 - v$ ja

$V_{300} = 300 + v - (300 - v) = 2v.$

Voitettujen kierrosten määrä voi olla mikä tahansa luku $v \in \{0, \dots, 300\}$, joten arvojoukko on $\{0, 2, 4, \dots, 600\}$

(c)
$$\mathbb{E}[V_{300}] = V_0 + \mathbb{E}\Big[2\sum_{s=1}^{300} \mathbb{1}_A(U_s)\Big] - \mathbb{E}[300] = 2\sum_{s=1}^{300} \underbrace{\mathbb{E}[\mathbb{1}_A(U_s)]}_{\mathbb{P}(U_s \in A) = \frac{18}{37}} = 600 \cdot \frac{18}{37} \approx 291.89$$

(d)
$$\Theta_s = \mathbb{1}_{\{U_s \in [1,18]\}} \overset{\text{H4T2}}{\Longrightarrow} \mathbb{P}(\Theta_s = i) = \begin{cases} \mathbb{P}(U_s \in [1,18]) & i = 1 \\ 1 - \mathbb{P}(U_s \in [1,18]) & i = 0 \end{cases} = \begin{cases} \frac{18}{37} & i = 1 \\ \frac{19}{37} & i = 0 \end{cases}$$

$\therefore\ \Theta_s \sim \text{Ber}(p)$, $p = \frac{18}{37}$

(e)
$$V_t = V_0 + \sum_{s=1}^{t} (2\mathbb{1}_A(U_s) - 1) = V_0 + \sum_{s=1}^{t} (2\mathbb{1}_{\{U_s \in A\}} - 1) = V_0 + 2\sum_{s=1}^{t} \Theta_s - t$$

$$= V_0 + g\Big(\sum_{s=1}^{t} \Theta_s\Big), \quad \text{missä } g(z) = 2z - t$$

(f) Olk $Z = \sum_{s=1}^{300} \Theta_s$

$$\mathbb{P}\Big(\frac{V_{300} - V_0}{V_0} \geq 0.1\Big) = \mathbb{P}(V_0 + g(Z) \geq V_0 \cdot 0.1) = \mathbb{P}(2Z - 300 \geq 30)$$

$$= \mathbb{P}(Z \geq 165) \underset{\substack{\uparrow \\ \text{H4T2} \\ Z \sim \text{Bin}(p, 300)}}{=} \sum_{k=165}^{300} \binom{300}{165} p^k (1-p)^{300-k}$$

5.2 $X \sim \text{Bin}(n, p)$

(a) $X = Z_1 + \dots + Z_n$, missä $Z_i \sim \text{Bernoulli}(p)$ ja $Z_i \perp\!\!\!\perp Z_j$ $i \neq j$

$$G_{Z_i}(t) = \mathbb{E}[t^{Z_i}] = t^0(1-p) + t^1 p = 1 - p + pt$$

$$G_X(t) = \mathbb{E}[t^X] = \mathbb{E}[t^{\sum_{i=1}^{n} Z_i}] = \mathbb{E}[t^{Z_1}] \cdots \mathbb{E}[t^{Z_n}] = (G_{Z_1}(t))^n = (1 - p + pt)^n$$

(b) $G_X(t)$ on määritelty kaikille $t \in \mathbb{R}$

(c)
$$G_X'(t) = n(1-p+pt)^{n-1} \cdot p = np(1-p+pt)^{n-1}$$

$$\mathbb{E}[X] = G_X'(1) = np(1 - p + p \cdot 1)^{n-1} = np$$

(d)
$$G_X''(t) \overset{n \geq 2}{=} n(n-1)p(1-p+pt)^{n-2}p = n(n-1)p^2(1-p+pt)^{n-2}$$

$$\text{var}(X) = G_X''(1) + G_X'(1) - (G_X'(1))^2 = n(n-1)p^2 + np - (np)^2$$

$$= (np)^2 - np^2 + np - (np)^2 = np(1-p)$$

5.3 $X \sim \text{Poisson}(\lambda)$

(a) $G_X(t) = \mathbb{E}[t^X] = \sum_{k=0}^{\infty} t^k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = e^{-\lambda} e^{\lambda t} = e^{\lambda(t-1)}$

(b) G_X on määritelty kaikilla $t \in \mathbb{R}$

(c) $G_X'(t) = \lambda e^{\lambda(t-1)}$, $G_X''(t) = \lambda^2 e^{\lambda(t-1)}$

$\mathbb{E}[X] = G_X'(1) = \lambda e^{\lambda(1-1)} = \lambda$

(d) $\text{var}(X) = G_X''(1) + G_X'(1) - (G_X'(1))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$

5.4 $X \sim \text{Pois}(\lambda_1)$, $Y \sim \text{Pois}(\lambda_2)$, $X \perp\!\!\!\perp Y$

$G_{X+Y}(t) \overset{\text{Esim 9.6}}{=} G_X(t) G_Y(t) \overset{5.3}{=} e^{\lambda_1(t-1)} e^{\lambda_2(t-1)} = e^{\lambda_1(t-1) + \lambda_2(t-1)}$

$= e^{(\lambda_1 + \lambda_2)(t-1)}$

G_{X+Y} on sama kuin $\text{Poisson}(\lambda_1 + \lambda_2)$ jakautuneen satunnaismuuttujan tngf, joten $X+Y$ on Poisson-jakautunut (LAUSEEN 9.4 perusteella tngf määrää jakauman 1-käsitteisesti)

5.5 $N, X_1, X_2, X_3, \ldots \perp\!\!\!\perp$ $\mathbb{Z}_+$-arvoisia, $X_i \overset{d}{=} X$ kaikilla i.

$M = \sum_{i=1}^{N} X_i$

$G_M(t) \overset{L\,9.9}{=} G_N(G_X(t))$

$G_M'(t) = G_N'(G_X(t)) G_X'(t)$

$G_M''(t) = G_N''(G_X(t))(G_X'(t))^2 + G_N'(G_X(t)) G_X''(t)$

(a) $\mathbb{E}[M] = G_M'(1) = G_N'(\underbrace{G_X(1)}_{=1}) G_X'(1) = \mathbb{E}[N]\mathbb{E}[X]$

(b) $\text{var}(M) = G_M''(1) + G_M'(1) - (G_M'(1))^2$

$= G_N''(1)(\mathbb{E}[X])^2 + \mathbb{E}[N] G_X''(1) + \mathbb{E}[N]\mathbb{E}[X] - (\mathbb{E}[N]\mathbb{E}[X])^2$

$G_N''(1) = \text{var}(N) + \mathbb{E}[N]^2 - \mathbb{E}[N]$

$G_X''(1) = \text{var}(X) + \mathbb{E}[X]^2 - \mathbb{E}[X]$

$= (\text{var}(N)(\mathbb{E}[X])^2 + (\mathbb{E}[N]\mathbb{E}[X])^2 - \mathbb{E}[N]\mathbb{E}[X]^2) - (\mathbb{E}[N]\mathbb{E}[X])^2$
$+ (\text{var}(X)\mathbb{E}[N] + \mathbb{E}[N]\mathbb{E}[X]^2 - \mathbb{E}[N]\mathbb{E}[X]) + \mathbb{E}[N]\mathbb{E}[X]$

$= \text{var}(N)(\mathbb{E}[X])^2 + \text{var}(X)\mathbb{E}[N]$

$\square$