University of Jyväskylä
Department of Mathematics and Statistics
MATA280 Foundations of stochastics

Exercise 4
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H. Seppälä / E. Laukkarinen

4.1 *Quick loans.* Let us consider the problem 6 of exercise 1. Calculate the following

(a) expected profit of Strata on condition that Antti pays back everything.

(b) expected profit of Strata on condition that both pay back something.

(c) standard deviation of the profit of Strata.

4.2 *Indicator random variable.* The *indicator* of an event $A \subset \Omega$ in a discrete probability space (Ω, P) is the $\{0,1\}$-valued random variable

$$1_A(\omega) = \begin{cases} 1, & \text{when } \omega \in A, \\ 0 & \text{else.} \end{cases}$$

(a) Find out the distribution of 1_A.

(b) Compute the expectation and variance of 1_A.

(c) Find out the distribution of $Z = 1_{A_1} + \cdots + 1_{A_n}$ when we assume that $1_{A_1}, 1_{A_2}, \ldots, 1_{A_n}$ are independent and $\mathbb{P}(A_i) = p$ for all $i = 1, ..., n$.

(d) Compute the expectation and variance of Z.

4.3 Let $X, X_1, X_2, ...$ be identically distributed independent random varibles. The *sample variance* is defined by formula

$$v = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2,$$

where $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ is the *sample mean.* Show that

$$\operatorname{Var}\left(\bar{X}\right) = \frac{\operatorname{Var}(X)}{n} \quad \text{and} \quad \mathbb{E}[v] = \operatorname{Var}(X).$$

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4.4 *Law of large number does not always hold.* Consider two independent collections of random variables such that

- $\{X_1, X_2, \dots\}$ are independent and identically distributed, and $\mathbb{E}(X_1) = 1$.
- $\{Y_1, Y_2, \dots\}$ are independent and identically distributed, and $\mathbb{E}(Y_1) = 2$.

Let us first flip a coin and then define

$$S_n = \begin{cases} X_1 + X_2 + \dots + X_n, & \text{if we get heads,} \\ Y_1 + Y_2 + \dots + Y_n, & \text{if we get tails.} \end{cases}$$

Prove that

(a) $\mathbb{E}(S_n/n) = 3/2$.

(b) $\mathbb{P}(|S_n/n - 3/2| > 1/4)$ does not converge to zero as n grows.

(c) Is the above observation in conflict with the law of large numbers?

4.5 Let X and Y random variables, with finite expected values. Show, that

(a) $\mathbb{E}\big[aX + bY\big] = a\mathbb{E}\big[X\big] + b\mathbb{E}\big[Y\big]$ and

(b) $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$,

whenever $a, b \in \mathbb{R}$.