A Fourier approach to pathwise stochastic integration
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**Abstract**

We show how Haar and Schauder functions and more generally Fourier analysis may be used to understand basic problems in stochastic analysis at depth. We start by a Haar-Schauder development of the Brownian motion, revealing its regularity properties. We prove that this development can be used for a simple and efficient proof of Schilder's theorem. Following [GIP15], we then show that it can be used on a pathwise level to explain Young's integral. Combined with the concept of controlled paths, it can be extended to provide the Stratonovich type integral of rough path analysis.

## 1 Ciesielski's isomorphism

We start by explaining Ciesielski's isomorphism between $\mathcal{C}^\alpha = C^\alpha([0,1],\mathbb{R}^d)$ and $\ell^\infty(\mathbb{R}^d)$ which is at the beginning of our approach. The *Haar functions* $(H_{pm}, p \geq 0, 1 \leq m \leq 2^p)$ are defined as $H_{00} \equiv 1$,

$$H_{pm}(t) := \begin{cases} \sqrt{2^p}, & t \in \left[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}\right), \\ -\sqrt{2^p}, & t \in \left[\frac{2m-1}{2^{p+1}}, \frac{m}{2^p}\right), \\ 0, & \text{otherwise.} \end{cases}$$

The Haar functions are a complete orthonormal system of $L^2([0,1],\mathrm{d}t)$. For convenience of notation, we also define $H_{p0} \equiv 0$ for $p \geq 1$. The primitives of the Haar functions are called *Schauder functions*. They are given by $G_{pm}(t) := \int_0^t H_{pm}(s)\mathrm{d}s$ for $t \in [0,1]$, $p \geq 0$, $0 \leq m \leq 2^p$. More explicitly, $G_{00}(t) = t$ and for $p \geq 1$, $1 \leq m \leq 2^p$

$$G_{pm}(t) = \begin{cases} 2^{p/2}\left(t - \frac{m-1}{2^p}\right), & t \in \left[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}\right), \\ -2^{p/2}\left(t - \frac{m}{2^p}\right), & t \in \left[\frac{2m-1}{2^{p+1}}, \frac{m}{2^p}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Since every $G_{pm}$ satisfies $G_{pm}(0) = 0$, we are only able to expand functions $f$ with $f(0) = 0$ in terms of this family $(G_{pm})$. Therefore, we complete $(G_{pm})$ once more, by defining $G_{-10}(t) := 1$ for all $t \in [0, 1]$. To abbreviate notation, we define the times $t^i_{pm}$, $i = 0, 1, 2$, as

$$t^0_{pm} := \frac{m-1}{2^p}, \quad t^1_{pm} := \frac{2m-1}{2^{p+1}}, \quad t^2_{pm} := \frac{m}{2^p},$$

for $p \geq 1$ and $1 \leq m \leq 2^p$. Further, we set $t^0_{-10} := 0$, $t^1_{-10} := 0$, $t^2_{-10} := 1$, and $t^0_{00} := 0$, $t^1_{00} := 1$, $t^2_{00} := 1$, as well as $t^i_{p0} := 0$ for $p \geq 1$ and $i = 0, 1, 2$. The definition of $t^i_{-10}$ and $t^i_{00}$ for $i \neq 1$ is rather arbitrary, but the definition for $i = 1$ simplifies for example the statement of Lemma 1.2 below.

For $f \in C([0, 1], \mathbb{R}^d)$, $p \in \mathbb{N}$, and $1 \leq m \leq 2^p$, we write

$$\begin{aligned}\langle H_{pm}, \mathrm{d}f\rangle :=& 2^{\frac{p}{2}} \left[\left(f\left(t^1_{pm}\right) - f\left(t^0_{pm}\right)\right) - \left(f\left(t^2_{pm}\right) - f\left(t^1_{pm}\right)\right)\right] \\ =& 2^{\frac{p}{2}} \left[2f\left(t^1_{pm}\right) - f\left(t^0_{pm}\right) - f\left(t^2_{pm}\right)\right]\end{aligned}$$

and $\langle H_{00}, \mathrm{d}f\rangle := f(1) - f(0)$ as well as $\langle H_{-10}, \mathrm{d}f\rangle := f(0)$. Note that we only defined $G_{-10}$ and not $H_{-10}$.

The norm $\|\cdot\|_\alpha$ is defined as

$$\|f\|_\alpha := \|f\|_\infty + \sup_{0 \leq s < t \leq 1} \frac{|f_{s,t}|}{|t-s|^\alpha},$$

where we introduced the notation

$$f_{s,t} := f(t) - f(s).$$

**Lemma 1.1** ([Cie60])**.** *Let $\alpha \in (0, 1)$. A continuous function $f : [0, 1] \to \mathbb{R}^d$ is in $\mathcal{C}^\alpha$ if and only if $\sup_{p,m} 2^{p(\alpha-1/2)}|\langle H_{pm}, \mathrm{d}f\rangle| < \infty$. In this case*

$$\sup_{p,m} 2^{p(\alpha-1/2)}|\langle H_{pm}, \mathrm{d}f\rangle| \simeq \|f\|_\alpha \text{ and} \tag{1}$$

$$\|f - f_{N-1}\|_\infty = \Big\| \sum_{p=N}^\infty \sum_{m=0}^{2^p} |\langle H_{pm}, \mathrm{d}f\rangle| G_{pm} \Big\|_\infty \lesssim \|f\|_\alpha 2^{-\alpha N}.$$

*Proof.* To estimate the development coefficients, fix $p \geq 1, 1 \leq m \leq 2^p$. We have

$$|\langle H_{pm}, \mathrm{d}f\rangle| \leq 2^{\frac{p}{2}+1}||f||_\alpha 2^{-\alpha(p+1)},$$

whereas, by disjointness of the supports of the Schauder functions of one dyadic generation we have

$$|| \sum_{m=1}^{2^p} G_{pm}||_\infty \leq 2^{-\frac{p}{2}}.$$

This implies the estimate for the (absolute) contribution of one generation

$$| \sum_{m=0}^{2^p} |\langle H_{pm}, \mathrm{d}f\rangle| G_{pm}| \lesssim ||f||_\alpha 2^{-\alpha(p+1)},$$

where $x \lesssim y$ means $x \leq Cy$ with a universal constant $C$. This is summable in $p$, and gives the desired estimate.

To prove the equivalence of norms, note that for $p \geq 1, 1 \leq m \leq 2^p$ we have $||G_{pm}||_\alpha = 2^{\frac{p}{2}} 2^{(-p-1)(1-\alpha)} = 2^{p(\alpha-\frac{1}{2})+\alpha-1}$, while $\langle H_{pm}, \mathrm{d}G_{pm}\rangle = \langle H_{pm}, H_{pm}\rangle = 1$. Hence

$$||G_{pm}||_\alpha \gtrsim 2^{p(\alpha-\frac{1}{2})} |\langle H_{pm}, \mathrm{d}G_{pm}\rangle|.$$

$\square$

**Lemma 1.2.** *For $f\colon [0,1] \to \mathbb{R}^d$, the function*

$$f_k := \langle H_{-10}, \mathrm{d}f\rangle G_{-10} + \langle H_{00}, \mathrm{d}f\rangle G_{00} + \sum_{p=0}^{k} \sum_{m=1}^{2^p} \langle H_{pm}, \mathrm{d}f\rangle G_{pm} = \sum_{p=-1}^{k} \sum_{m=0}^{2^p} \langle H_{pm}, \mathrm{d}f\rangle G_{pm}$$

*is the linear interpolation of $f$ between the points $t^1_{-10}, t^1_{00}, t^1_{pm}$, $0 \leq p \leq k, 1 \leq m \leq 2^p$. If $f$ is continuous, then $(f_k)$ converges uniformly to $f$ as $k \to \infty$.*

*Proof.* Let $g_k$ be the linear interpolation of $f$ between the points $t^1_{-10}, t^1_{00}, t^1_{pm}$, $0 \leq p \leq k, 1 \leq m \leq 2^p$. Then obviously $g_k \in \mathcal{C}^\alpha$. Now $g_k - f_k = 0$, since by Lemma 1.2 $f_n \to g_k$ as $n \to \infty$, but by definition of $G_{pm}$ the contributions of dyadic generations bigger than $k$ have to vanish at the points $t^1_{-10}, t^1_{00}, t^1_{pm}$, $0 \leq p \leq k, 1 \leq m \leq 2^p$. The claimed convergence follows from uniform continuity on $[0,1]$. $\square$

**Theorem 1.3** ([Cie60])**.** *Let $0 < \alpha < 1$. For $p \geq 0, 1 \leq m \leq 2^p$ let*

$$c_{pm}(\alpha) = 2^{p(\alpha-\frac{1}{2})+\alpha-1}, \quad c_{p0}(\alpha) = 1, \quad c_{-10}(\alpha) = 1.$$

*Define*

$$\begin{aligned} T_\alpha \quad &: \quad \mathcal{C}^\alpha \to l^\infty(\mathbb{R}^d) \\ f \quad &\mapsto \quad (c_{-10}(\alpha)\langle H_{-10}, \mathrm{d}f\rangle, c_{00}\langle H_{00}, \mathrm{d}f\rangle, (c_{pm}(\alpha)\, \langle H_{pm}, \mathrm{d}f\rangle)_{p\geq 1, 1\leq m \leq 2^p}). \end{aligned}$$

*Then*

$$\begin{aligned} T_\alpha^{-1} \quad &: \quad l^\infty(\mathbb{R}^d) \to \mathcal{C}^\alpha \\ (\eta_{-10}, \eta_{00}, (\eta_{pm})_{p\geq 1, 1 \leq m \leq 2^p}) \quad &\mapsto \quad \eta_{-10} G_{-10} + \eta_{00} G_{00} + \sum_{p=1}^{\infty} \sum_{1 \leq m \leq 2^p} \frac{1}{c_{pm}(\alpha)} \eta_{pm}\, G_{pm}. \end{aligned}$$

*$T_\alpha$ is an isomorphism, and for the operator norms we have the following inequalities*

$$||T_\alpha|| = 1, \quad ||T_\alpha^{-1}|| \leq \frac{2}{(2^\alpha - 1)(2^{1-\alpha}-1)}.$$

*Proof.* By definition, for $p \geq 1, 1 \leq m \leq 2^p$ we have

$$|\langle H_{pm}, \mathrm{d}f\rangle| \leq 2^{-(p+1)\alpha + \frac{p}{2}+1} ||f||_\alpha = \frac{1}{c_{pm}(\alpha)} ||f||_\alpha. \tag{2}$$

Therefore, $T_\alpha$ is well defined, and we have

$$||T_\alpha|| \leq 1.$$

The equation $||T_\alpha|| = 1$ follows from the proof of Lemma 1.1. This Lemma also shows that $T_\alpha$ is one-to-one and that $T_\alpha^{-1}$ is its inverse.

We next prove the inequality for the operator norm of $T_\alpha^{-1}$.
Let $\eta = (\eta_{-10}, \eta_{00}, (\eta_{pm})_{p\geq 1, 1\leq m\leq 2^p}) \in l^\infty(\mathbb{R}^d)$ be given, choose $0 \leq s < t \leq 1$, and write $f = T_\alpha^{-1}(\eta)$. Then we have

$$|f(t) - f(s)| \leq ||\eta||_\infty [|t-s| + \sum_{p=1}^\infty \sum_{m=1}^{2^p} \frac{1}{c_{pm}(\alpha)} |G_{pm}(t) - G_{pm}(s)|]. \tag{3}$$

Now choose $p_0 \geq 1$ such that

$$2^{-p_0-1} < |t-s| \leq 2^{-p_0}.$$

Then for $1 \leq p < p_0$ due to the fact that the supports of $G_{pm}, p \geq 1, 1 \leq m \leq 2^p$ are disjoint

$$\begin{aligned} & \sum_{m=1}^{2^p} \frac{1}{c_{pm}(\alpha)} |G_{pm}(t) - G_{pm}(s)| \qquad (4)\\ = & \sup_{1\leq m\leq 2^p} \frac{1}{c_{pm}(\alpha)} |G_{pm}(t) - G_{pm}(s)| \\ \leq & \; 2^{-p(\alpha-\frac{1}{2})-\alpha+1}\, 2^{\frac{p}{2}} \; |t-s| \\ \leq & \; 2^{p(1-\alpha)-\alpha+1-p_0(1-\alpha)} \; |t-s|^\alpha = (2^{1-\alpha})^{(1+p-p_0)} \; |t-s|^\alpha, \end{aligned}$$

while for $p \geq p_0$

$$\begin{aligned} & \sum_{m=1}^{2^p} \frac{1}{c_{pm}(\alpha)} |G_{pm}(t) - G_{pm}(s)| \qquad (5)\\ \leq & \; 2^{-p(\alpha-\frac{1}{2})-\alpha+1}\; 2^{-\frac{p}{2}-1} \\ \leq & \; 2^{-p\alpha-\alpha+(p_0+1)\alpha} \; |t-s|^\alpha = (2^\alpha)^{(p_0-p)} \; |t-s|^\alpha. \end{aligned}$$

Combining (3), (4) and (5), we obtain the estimate

$$\frac{|f(t)-f(s)|}{|t-s|^\alpha} \leq \frac{2}{(2^\alpha-1)(2^{1-\alpha}-1)} ||\eta||_\infty,$$

and therefore

$$||T_\alpha^{-1}|| \leq \frac{2}{(2^\alpha-1)(2^{1-\alpha}-1)}.$$

$\square$

We can extend the isomorphism of Theorem 1.3 to subspaces of Hölder continuous functions which will arise later in the study of the LDP for Brownian motion. For $0 < \alpha \leq 1$ let $\mathcal{C}_0^\alpha$ be the subspace of $\mathcal{C}([0,1])$ composed of all functions $f$ for which $f(0) = 0$ and

$$\lim_{\delta\to 0} \sup_{0\leq s<t\leq 1, |t-s|\leq\delta} \frac{|f(t)-f(s)|}{|t-s|^\alpha} = 0.$$

The isomorphism of Theorem 1.3 will then be restricted to the subspace $\mathcal{C}_0$ of all sequences $\eta = (\eta_{pm})_{p\geq 1, 1\leq m\leq 2^p}$ in $\mathcal{C}$ which converge to 0 as $p \to \infty$ as a target space. The following Theorem holds, with a slightly, not essentially different proof.

**Theorem 1.4.** *Let $0 < \alpha < 1$. Let $c_{-10}, c_{00}, c_{pm}(\alpha), p \geq 1, 1 \leq m \leq 2^p$, be defined as in Theorem 1.3. Define*

$$\begin{aligned} T_{\alpha,0} \quad &: \quad \mathcal{C}_0^\alpha \to \mathcal{C}_0, \\ f \quad &\mapsto \quad (c_{-10}(\alpha)\langle H_{-10}, \mathrm{d}f\rangle, c_{00}\langle H_{00}, \mathrm{d}f\rangle, (c_{pm}(\alpha)\,\langle H_{pm}, df\rangle)_{p\geq 1, 1\leq m\leq 2^p}). \end{aligned}$$

*Then*

$$\begin{aligned} T_{\alpha,0}^{-1} \quad &: \quad \mathcal{C}_0 \to \mathcal{C}_0^\alpha, \\ ((\eta_{-10}, \eta_{00}, (\eta_{pm})_{p\geq 1, 1\leq m\leq 2^p}) \quad &\mapsto \quad \eta_{-10}G_{-10} + \eta_{00}G_{00} + \sum_{p=1}^\infty \sum_{1\leq m\leq 2^p} \frac{1}{c_{pm}(\alpha)}\eta_{pm}\, G_{pm}. \end{aligned}$$

*$T_{\alpha,0}$ is an isomorphism, and for the operator norms we have the following inequalities*

$$||T_{\alpha,0}|| = 1, \quad ||T_{\alpha,0}^{-1}|| \leq \frac{2}{(2^\alpha - 1)(2^{1-\alpha} - 1)}.$$

## 2 The Schauder representation of Brownian motion

We shall now present an approach of the study of one-dimensional Brownian motion which is close to Wiener's representation of Brownian motion by Fourier series with trigonometric functions as a basis. So in this section we set $d = 1$. Our basis will be given by the Haar-Schauder system of the preceding section. In fact, the trajectories of Brownian motion will be described just as in the preceding section continuous functions were isomorphically described by sequences. Given a Brownian motion $X$ indexed by the unit interval, with the same notation as in the preceding section we write it sample by sample as a series with coefficients $\langle H_{00}, \mathrm{d}X\rangle, \langle H_{pm}, \mathrm{d}X\rangle, p \geq 1, 1 \leq m \leq 2^p$. Due to the scaling properties and the structure of Haar functions, these random coefficients are i.i.d standard normal random variables. This, in turn, allows us to construct Brownian motion indexed by the unit interval by taking any sequence of i.i.d. standard normal variables $(Z_{00}, (Z_{pm})_{p\geq 1, 1\leq m\leq 2^p})$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and defining the stochastic process

$$W_t = Z_{00}G_{00}(t) + \sum_{p=1}^\infty \sum_{1\leq m\leq 2^p} Z_{pm}G_{pm}(t), \quad t \in [0,1]. \tag{6}$$

To get information about the quality of convergence of this Fourier series, we need to control the size of the random sequence $(Z_{pm})_{p\geq 1, 1\leq m\leq 2^p}$ in the following Lemma.

**Lemma 2.1.** *There exists a real valued random variable $C$ such that for $p \geq 1, 1 \leq m \leq 2^p$ we have*

$$|Z_{pm}| \leq C\,\sqrt{p \ln 2}. \tag{7}$$

*Proof.* For $x \geq 1, p \geq 1, 1 \leq m \leq 2^p$ we have

$$\mathbb{P}(|Z_{pm}| \geq x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du \leq \sqrt{\frac{2}{\pi}} \int_x^\infty u e^{-\frac{u^2}{2}} du = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}.$$

Hence for $\beta > 1$

$$\mathbb{P}(|Z_{pm}| \geq \sqrt{2\beta \ln 2^p}) \leq \sqrt{\frac{2}{\pi}}\, e^{-\beta \ln 2^p} = \sqrt{\frac{2}{\pi}} 2^{-\beta p}.$$

Therefore, the lemma of Borel-Cantelli yields that $|Z_{pm}| \leq \sqrt{4\beta p \ln 2}$ for almost all $p \geq 1, 1 \leq m \leq 2^p$ with probability 1. Hence the random variable $C = \sup_{p\geq 1, 1\leq m\leq 2^p} \frac{|Z_{pm}|}{\sqrt{p \ln 2}}$ is almost surely finite, and yields the desired inequality. □

The preceding Lemma enables us to state that the convergence in (6) is absolute and therefore the process continuous. Its law has the characteristics of the law of a Brownian motion, as the following Theorem shows.

**Theorem 2.2.** *The series in (6) converges absolutely in the uniform norm to a continuous process $W$ which is a Brownian motion on* $[0, 1]$.

*Proof.* Let us first prove the absolute convergence of the series in the uniform norm. This will evidently imply that $W$ is continuous. Let $p, q \geq 1$ be such that $q \geq p$. Then for $t \in [0, 1]$ we have with the random variable $C$ of the preceding Lemma

$$\begin{aligned} \sum_{n=p}^{q} \sum_{1\leq m\leq 2^n} |Z_{nm}| G_{nm}(t) &\leq C \sum_{n=p}^{q} \sum_{1\leq m\leq 2^n} \sqrt{n \ln 2}\; G_{nm}(t) \\ &\leq C \sum_{n=p}^{\infty} \sum_{1\leq m\leq 2^p} \sqrt{n}\; G_{nm}(t) \\ &\leq C \sum_{n=p}^{\infty} \sqrt{n}\; 2^{-\frac{n}{2}-1}, \end{aligned}$$

which converges to 0 as $p$ tends to $\infty$, independently of $t \in [0, 1]$.

To prove that $W$ is a Gaussian process with $\mathbb{E}(W_t) = 0$ and $\operatorname{cov}(W_t, W_s) = s \wedge t$, for $0 \leq s, t \leq 1$, we first note that the series also converges in square norm. In fact, we have for $t \in [0, 1]$, $p, q \geq 1$ such that $q \geq p$ by the law properties of $Z_{pm}, p \geq 1, 1 \leq m \leq 2^p$,

$$\mathbb{E}([\sum_{n=p}^{q} \sum_{1\leq m\leq 2^n} Z_{nm} G_{nm}(t)]^2) = \sum_{n=p}^{q} \sum_{1\leq m\leq 2^n} G_{nm}(t)^2 \leq \sum_{n=p}^{\infty} 2^{-n-2},$$

which converges to 0 as $p \to \infty$. Next, let $d \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_d \leq 1$, and $\theta = (\theta_1, \cdots, \theta_d) \in \mathbb{R}^d$ be given. We compute the Fourier transform $\varphi(\theta)$ of the vector $(W_{t_1}, \cdots, W_{t_d})$ at $\theta$. We have by dominated convergence and the law properties of $Z_{pm}, p \geq 1, 1 \leq m \leq 2^p$, again

$$\begin{aligned}
\varphi(\theta) &= \mathbb{E}(\exp(i\sum_{j=1}^d \theta_j W_{t_j})) \\
&= \mathbb{E}(\exp(i\sum_{j=1}^d \theta_j \sum_{p=0}^\infty \sum_{0\leq m\leq 2^p} Z_{pm} G_{pm}(t_j))) \\
&= \prod_{p=0}^\infty \prod_{0\leq m\leq 2^p} \mathbb{E}(\exp(iZ_{pm}\sum_{j=1}^d \theta_j G_{pm}(t_j))) \\
&= \prod_{p=0}^\infty \prod_{0\leq m\leq 2^p} \exp(-\frac{1}{2}(\sum_{j=1}^d \theta_j G_{pm}(t_j))^2) \\
&= \exp(-\frac{1}{2}\sum_{p=0}^\infty \sum_{0\leq m\leq 2^p} (\sum_{j=1}^d \theta_j G_{pm}(t_j))^2) \\
&= \exp(-\frac{1}{2}\sum_{j,k=1}^d \theta_j\theta_k \sum_{p=0}^\infty \sum_{0\leq m\leq 2^p} G_{pm}(t_j)G_{pm}(t_k)).
\end{aligned}$$

Now observe that Parseval's equation implies for $1 \leq j, k \leq d$

$$t_j \wedge t_k = \langle 1_{[0,t_j]}, 1_{[0,t_k]}\rangle = \sum_{p=0}^\infty \sum_{0\leq m\leq 2^p} \langle 1_{[0,t_j]}, H_{pm}\rangle \langle 1_{[0,t_k]}, H_{pm}\rangle = \sum_{p=0}^\infty \sum_{0\leq m\leq 2^p} G_{pm}(t_j)G_{pm}(t_k).$$

Therefore we finally obtain

$$\varphi(\theta) = \exp(-\frac{1}{2}\sum_{j,k=1}^d \theta_j\theta_k\ t_j \wedge t_k).$$

But this means that $(W_{t_1}, \cdots, W_{t_d})$ is Gaussian with expectation vector 0 and covariance matrix $C$ with entries $c_{jk} = t_j \wedge t_k, 1 \leq j, k \leq d$. It is easy to see that these properties imply that the process $W$ possesses independent increments which are Gaussian with mean 0 and variance corresponding to the length of the increment intervals. This, however, characterizes a Brownian motion. □

We now use the Schauder representation of Brownian motion to show its Hölder continuity properties.

**Theorem 2.3.** *The Brownian motion $W = (W_t)_{0\leq t\leq 1}$ is Hölder continuous of order $\alpha < 1/2$. Its trajectories are a.s. nowhere Hölder continuous of order $\alpha > 1/2$. Moreover we have (Lévy's modulus of continuity)*

$$\mathbb{P}\Big(\sup_{0\leq s<t\leq 1} \frac{|W_t - W_s|}{h(|t-s|)} < \infty\Big) = 1, \tag{8}$$

*where $h(u) = \sqrt{u\log(1/u)}, u > 0$. In particular, for $\alpha < \frac{1}{2}$, the trajectories of $W$ are $\mathbb{P}$-a.s. contained in the space $\mathcal{C}^\alpha$.*

*Proof.* Let first $\alpha \in ]0,1[$, $(c_{pm})_{p\geq 1, 1\leq m\leq 2^p}$ be a sequence of real numbers for which there exists $c \in \mathbb{R}$ such that for $p \geq 0, 1 \leq m \leq 2^p$ we have

$$|c_{pm}| \leq c\sqrt{p}.$$

Let

$$f(t) = \sum_{p=1}^{\infty} \sum_{1\leq m\leq 2^p} c_{pm}\, G_{pm}(t), \quad t \in [0,1].$$

We shall prove that $\sup_{0\leq s<t\leq 1} \frac{|f(t)-f(s)|}{h(|t-s|)} < \infty$. Due to Lemma 2.1, this will imply the claimed formula. In fact, due to the continuity properties of $G_{00}$, we may assume that $c_{00} = 0$. Then for $0 \leq s < t \leq 1$

$$|f(t) - f(s)| \leq \sum_{p=1}^{\infty} \sum_{m=1}^{2^p} |c_{pm}|\, |G_{pm}(t) - G_{pm}(s)|. \tag{9}$$

Now choose $p_0 \geq 1$ such that

$$2^{-p_0-1} < |t-s| \leq 2^{-p_0}.$$

Then for $1 \leq p < p_0$

$$\begin{aligned} &\sum_{m=1}^{2^p} |c_{pm}|\, |G_{pm}(t) - G_{pm}(s)| \qquad (10)\\ &= \sup_{1\leq m\leq 2^p} |c_{pm}|\, |G_{pm}(t) - G_{pm}(s)| \\ &\leq c\sqrt{p}\, 2^{\frac{p}{2}}\, |t-s| \\ &\leq c\sqrt{p}\, 2^{\frac{p-p_0}{2}}\, |t-s|^{\frac{1}{2}} \\ &\leq \frac{c}{\sqrt{\ln 2}} \sqrt{\frac{p}{p_0}}\, 2^{\frac{p-p_0}{2}} \sqrt{|t-s|\, \ln \frac{1}{|t-s|}}, \end{aligned}$$

while for $p \geq p_0$

$$\begin{aligned} &\sum_{m=1}^{2^p} |c_{pm}|\, |G_{pm}(t) - G_{pm}(s)| \qquad (11)\\ &\leq c\sqrt{p}\, 2^{-\frac{p}{2}} \\ &\leq \frac{c}{\sqrt{\ln 2}} \sqrt{\frac{p}{p_0}}\, 2^{\frac{p_0+1-p}{2}} \sqrt{|t-s|\, \ln \frac{1}{|t-s|}}. \end{aligned}$$

It is easy to see, for instance by estimating $\int_a^\infty \sqrt{x} 2^{-x} dx$, and $\int_1^a \sqrt{x} 2^x dx$ for $a \geq 1$ using integration by parts that the sum in $p$ of the two estimates can be taken and yields a finite upper bound which does not depend on $p_0$. Hence (9), (10) and (11) imply

$$\frac{|f(t) - f(s)|}{\sqrt{|t-s|\, \ln \frac{1}{|t-s|}}} \leq c',$$

for some constant $c'$ independent of $s$ and $t$. This implies the desired inequality, and all claims about Hölder continuity for $\alpha < \frac{1}{2}$.

Let us next fix $\alpha > \frac{1}{2}$. For $c > 0, \epsilon > 0$ let

$$\Gamma(\alpha, c, \epsilon) = \{\omega \in \Omega :\ \exists\, s \in [0,1]\ \forall\, t \in [0,1],\ |s-t| \le \epsilon :\ |W_t(\omega) - W_s(\omega)| \le c|s-t|^\alpha\}.$$

We will show that $\mathbb{P}(\Gamma(\alpha, c, \epsilon)) = 0$ for all $c, \epsilon > 0$, and thus that $W$ is a. s. nowhere Hölder continuous of order $\alpha$. To this end, for all $m, n \in \mathbb{N}, m \le n$, and $0 \le k < n$ let

$$X_{m,k} = \max\{|W_{\frac{j}{n}} - W_{\frac{j+1}{n}}| : k \le j < m+k\}.$$

Let $\omega \in \Gamma(\alpha, c, \epsilon)$. Choose $n \in \mathbb{N}$ so that $\frac{m}{n} \le \epsilon$. Let $s \in [0,1]$ be given such that for all $t \in [0,1]$ satisfying $|s-t| \le \epsilon$ we have $|W_t(\omega) - W_s(\omega)| \le c|s-t|^\alpha$. Choose $0 \le k \le n-m$ such that $\frac{k}{n} \le s < \frac{k+m}{n}$. Then for $k \le j < k+m$

$$\begin{aligned} |W_{\frac{j}{n}}(\omega) - W_{\frac{j+1}{n}}(\omega)| &\le |W_{\frac{j}{n}}(\omega) - W_s(\omega)| + |W_s(\omega) - W_{\frac{j+1}{n}}(\omega)| \\ &\le c|\frac{j}{n} - s|^\alpha + c|s - \frac{j+1}{n}|^\alpha \le 2c(\frac{m}{n})^\alpha. \end{aligned}$$

This proves that

$$\Gamma(\alpha, c, \epsilon) \subset \{\min_{0 \le k \le n-m} X_{m,k} \le 2c(\frac{m}{n})^\alpha\}. \tag{12}$$

Let us now estimate the probability of the latter set. Indeed, we have using independence and stationarity of the laws of the increments of $W$, and its scaling properties

$$\begin{aligned} \mathbb{P}(\min_{0 \le k \le n-m} X_{m,k} \le 2c(\frac{m}{n})^\alpha) &\le n\mathbb{P}(X_{m,1} \le 2c(\frac{m}{n})^\alpha) \\ &\le n\mathbb{P}(|W_{\frac{1}{n}}| \le 2c(\frac{m}{n})^\alpha)^m \\ &= n\mathbb{P}(|W_1| \le 2c\sqrt{n}(\frac{m}{n})^\alpha)^m \\ &\le n[\sqrt{\frac{2}{\pi}} 2c\sqrt{n}(\frac{m}{n})^\alpha]^m = n^{1+(\frac{1}{2}-\alpha)m}[\sqrt{\frac{2}{\pi}} 2cm^\alpha]^m. \end{aligned}$$

Now choose $m$ so that $1 + (\frac{1}{2} - \alpha)m < 0$. Then let $n \to \infty$ to obtain that

$$\mathbb{P}(\Gamma(\alpha, c, \epsilon)) = 0,$$

as desired. □

# 3 Large deviations for Brownian motion

In this section, we shall apply the Haar-Schauder expansion of Brownian motion derived in the preceding section to show that it gives easy access to the LDP for Brownian motion, usually comprised in Schilder's theorem. In fact we will establish how the expansion allows to reduce the calculation of the rate functions to the ones for simple one-dimensional Gaussian variables.

### 3.1 Large deviations for one-dimensional Gaussian random variables

The large deviation rate for a one-dimensional Gaussian unit random variable can be directly calculated. Consider a random variable $Z$ with standard normal law, and let $\mu_\epsilon$ be the law of $\sqrt{\epsilon}Z$. Then the following statement holds.

**Theorem 3.1.** *Let*

$$I(x) = \frac{x^2}{2}, \quad x \in \mathbb{R}.$$

*Then for any open set $G \subset \mathbb{R}$ and any closed set $F \subset \mathbb{R}$ we have*

$$-\inf_{x\in G} I(x) \le \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(G),$$

$$-\inf_{x\in F} I(x) \ge \limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(F).$$

*Proof.* We argue for a closed set $F \subset \mathbb{R}$. Let $a = \inf\{|x| : x \in F\}$. Note that the case $a = 0$ is trivial. We may therefore assume that $a > 0$. By symmetry we may further assume that there exists $b \ge a$ such that $F \subset ]-\infty, -b] \cup [a, \infty[$. Hence for $\epsilon > 0$

$$\mu_\epsilon(F) \le \mu_\epsilon([a,\infty[) + \mu_\epsilon(]-\infty,-b]) \le \frac{2}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{\epsilon}}}^{\infty} \exp(-\frac{x^2}{2})dx.$$

For $u > 1$ we have

$$\int_u^\infty \exp(-\frac{x^2}{2})dx \le \int_u^\infty x\exp(-\frac{x^2}{2})dx = \exp(-\frac{1}{2}u^2).$$

Hence for $\epsilon < a^2$

$$\epsilon \ln \mu_\epsilon(F) \le \epsilon[\ln(\frac{2}{\sqrt{2\pi}}) - \frac{a^2}{2\epsilon}],$$

and therefore

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(F) \le -\frac{a^2}{2} = -\inf_{x\in F} I(x).$$

For open sets we need a different inequality. In fact, integration by parts gives for $u > 1$

$$\int_u^\infty \exp(-\frac{x^2}{2})dx = \frac{1}{u}\exp(-\frac{1}{2}u^2) - \int_u^\infty \frac{1}{x^2}\exp(-\frac{x^2}{2})dx,$$

hence

$$\frac{1}{u}\exp(-\frac{1}{2}u^2) \le (1 + \frac{1}{u^2})\int_u^\infty \exp(-\frac{x^2}{2})dx$$

and

$$\frac{u}{1+u^2}\exp(-\frac{1}{2}u^2) \le \int_u^\infty \exp(-\frac{x^2}{2})dx.$$

Now let $G \subset \mathbb{R}$ be open, $y \in G$. By symmetry, we may assume that $y > 0$. Let, moreover, $a, b > 0$ such that $y \in ]a, b[ \subset G$. Then, for $\epsilon$ small enough we have

$$\begin{aligned} \mu_\epsilon(G) &\ge \mu_\epsilon(]a,\infty[) - \mu_\epsilon([b,\infty[) \ge \frac{1}{\sqrt{2\pi}}[\frac{\frac{a}{\sqrt{\epsilon}}}{1+\frac{a^2}{\epsilon}}\exp(-\frac{a^2}{2\epsilon}) - \exp(-\frac{b^2}{2\epsilon})] \\ &\ge \frac{1}{\sqrt{2\pi}}\frac{\frac{a}{2\sqrt{\epsilon}}}{1+\frac{a^2}{\epsilon}}\exp(-\frac{a^2}{2\epsilon}). \end{aligned}$$

Therefore

$$\liminf_{\epsilon\to 0} \epsilon \ln \mu_\epsilon(G) \geq -\frac{a^2}{2} \geq -\frac{y^2}{2} = -I(y).$$

This implies the lower bound.

□

### 3.2 Large deviations for one-dimensional Brownian motion in Hölder space

In this section we use the Fourier series decomposition of one-dimensional Brownian motion in Hölder space $\mathcal{C}_0^\alpha$ of order $\alpha < \frac{1}{2}$, and Ciesielski's isomorphism mapping this space to a sequence space, to calculate the rate function arising in LDP for Brownian motion, as for instance specified in Schilder's theorem. This remarkable approach was presented in Baldi and Roynette [BR92]. Again we have $d = 1$. Let $W$ be a one-dimensional Brownian motion indexed by $[0, 1]$, described by

$$W = Z_{00}G_{00}(t) + \sum_{p=1}^{\infty} \sum_{1\leq m\leq 2^p} Z_{pm}\, G_{pm},$$

with a sequence $(Z_{00}, (Z_{pm})_{p\geq 1, 1\leq m\leq 2^p})$ of i.i.d standard normal variables, and the Schauder functions $(G_{pm})_{p\geq 0, 0\leq m\leq 2^p}$, as described in the previous section. Recall the Haar functions $(H_{pm})_{p\geq 0, 0\leq m\leq 2^p}$ and the sequences $(c_{-10}, c_{00}, (c_{pm}(\alpha))_{p\geq 1, 1\leq m\leq 2^p})$ appearing in Ciesielski's isomorphism in Theorem 1.3 for $0 < \alpha < 1$, given by

$$c_{pm}(\alpha) = 2^{p(\alpha-\frac{1}{2})+\alpha-1}, \quad c_{00}(\alpha) = 1, \quad c_{-10}(\alpha) = 1, \tag{13}$$

if $p \geq 1, 1 \leq m \leq 2^p$. We investigate the asymptotic behavior of the family of probability measures $(\mu_\epsilon)_{\epsilon>0}$, where $\mu_\epsilon$ is the law of $\sqrt{\epsilon}W$, $\epsilon > 0$. We remark that according to Theorem 2.3 for any $\epsilon > 0, 0 < \alpha < \frac{1}{2}$ we have

$$\mu_\epsilon(\mathcal{C}_0^\alpha) = 1. \tag{14}$$

The large deviation rates for Brownian motion will crucially depend on the following function space, the *Cameron-Martin space* of absolutely continuous functions.

**Definition 3.2.** Let

$$\begin{aligned} \mathcal{H}_1 &= \Big\{ f : [0,1] \to \mathbb{R},\ f(0) = 0,\ f \text{ abs. cont. with density } \dot{f} \in L^2([0,1]) \Big\} \\ &= \Big\{ \int_0^t \dot{f}(s)ds,\ \dot{f} \in L^2([0,1]) \Big\}. \end{aligned} \tag{15}$$

By means of (15) we can define the rate function for Brownian motion.

**Definition 3.3.** Let

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 (\dot{f})^2(u)du, & \text{if } f \in \mathcal{H}_1, \\ \infty, & \text{otherwise.} \end{cases} \tag{16}$$

In the following Theorem the rate function for an LDP for Brownian motion is calculated for basic sets of a topology that is finer than the supremum norm topology usually employed on Wiener space. Using standard arguments, it can be enhanced to an LDP in Schilder's Theorem. We consider the following basic sets of the Hölder topology. For $\delta > 0, \psi \in \mathcal{C}_0^\alpha$ denote $B_\delta^\alpha(\psi) = \{f \in \mathcal{C}_0^\alpha : ||f - \psi||_\alpha < \delta\}$.

**Theorem 3.4.** *Let $0 < \alpha < 1/2$, $\delta > 0$ and $\psi \in (\mathcal{C}_0^\alpha, \|\cdot\|_\alpha)$. Then with the rate function $I$ defined by (16)*

$$\lim_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(B_\delta^\alpha(\psi)) = - \inf_{f \in B_\delta^\alpha(\psi)} I(f), \tag{17}$$

$$\lim_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(\overline{B_\delta^\alpha(\psi)}) = - \inf_{f \in \overline{B_\delta^\alpha(\psi)}} I(f). \tag{18}$$

*Proof.* We give the arguments for (17). The proof of (18) is almost identical.

1. We use the Schauder representation of Brownian motion $W$ and the function $\psi$ given by

$$W = Z_{00}G_{00} + \sum_{p \geq 1, 1 \leq m \leq 2^p} Z_{pm}G_{pm} \quad \text{and} \quad \psi = \xi_{00}G_{00} + \sum_{p \geq 1, 1 \leq m \leq 2^p} \frac{\xi_{pm}}{c_{pm}(\alpha)} G_{pm}. \tag{19}$$

We recall the inverse of (the restriction of) Ciesielski's isomorphism

$$T_{\alpha,0}^{-1} : l^\infty(\mathbb{R}) \to \mathcal{C}_0^\alpha, \quad (\eta_{00}, (\eta_{pm})_{p \geq 0, 1 \leq m \leq 2^p}) \mapsto \eta_{00}G_{00} + \sum_{p=1}^{\infty} \sum_{1 \leq m \leq 2^p} \frac{1}{c_{pm}(\alpha)} \eta_{pm}\ G_{pm},$$

and remark that the sequence $(\xi_{00}, (\xi_{pm})_{p \geq 1, 1 \leq m \leq 2^p})$ in the representation of $\psi$ just satisfies $T_{\alpha,0}(\psi) = (\xi_{00}, (\xi_{pm})_{p \geq 1, 1 \leq m \leq 2^p})$, while $T_{\alpha,0}(\sqrt{\epsilon}W) = (\sqrt{\epsilon}Z_{00}, \sqrt{\epsilon}c_{pm}(\alpha)Z_{pm})_{p \geq 1, 1 \leq m \leq 2^p}$. We therefore have, denoting for notational simplicity, $Z_{p0} = 0, \xi_{p0} = 0, c_{p0} = 1, p \geq 1$,

$$\sqrt{\varepsilon}W \in B_\delta^\alpha(\psi) \iff \sup_{p \geq 0, 0 \leq m \leq 2^p} |\sqrt{\varepsilon}c_{pm}(\alpha)Z_{pm} - \xi_{pm}| < \delta.$$

Hence

$$(\sqrt{\varepsilon}W)^{-1}[B_\delta^\alpha(\psi)] = \bigcap_{p \geq 0, 0 \leq m \leq 2^p} \Big\{ \sqrt{\varepsilon}c_{pm}(\alpha)Z_{pm} \in ]\xi_{pm} - \delta, \xi_{pm} + \delta[ \Big\}.$$

Since $(Z_{pm})_{p \geq 0, 0 \leq m \leq 2^p}$ is a family of independent random variables, we obtain for $\epsilon > 0$

$$\mu_\epsilon(B_\delta^\alpha(\psi)) = \prod_{p \geq 0, 0 \leq m \leq 2^p} \mathbb{P}\Big( \sqrt{\varepsilon}c_{pm}(\alpha)Z_{pm} \in ]\xi_{pm} - \delta, \xi_{pm} + \delta[ \Big) = \prod_{p \geq 0, 0 \leq m \leq 2^p} P_{pm}(\varepsilon). \tag{20}$$

We split the sequence of probabilities $(P_{pm}(\varepsilon))_{p \geq 0, 0 \leq m \leq 2^p}$ into four different parts to be treated separately:

$$\begin{aligned}
\Lambda_1 &= \Big\{ (p, m) : p \geq 0, 0 \leq m \leq 2^p,\ 0 \notin [\xi_{pm} - \delta, \xi_{pm} + \delta] \Big\}, \\
\Lambda_2 &= \Big\{ (p, m) : p \geq 0, 0 \leq m \leq 2^p,\ \xi_{pm} = \pm\delta \Big\}, \\
\Lambda_3 &= \Big\{ (p, m) : p \geq 0, 0 \leq m \leq 2^p, [\xi_{pm} - \delta, \xi_{pm} + \delta] \supset \Big[ -\tfrac{\delta}{2}, \tfrac{\delta}{2} \Big] \Big\}, \\
\Lambda_4 &= (\Lambda_3)^c \setminus (\Lambda_1 \cup \Lambda_2).
\end{aligned}$$

Let us recall that $(\xi_{pm})_{p \geq 0, 0 \leq m \leq 2^p} \in l^\infty(\mathbb{R})$, so $\Lambda_3$ contains almost all $(p, m), p \geq 0, 0 \leq m \leq 2^p$, and hence $\Lambda_1 \cup \Lambda_2 \cup \Lambda_4 = (\Lambda_3)^c$ is finite.

2. Let us first discuss the contribution of $\Lambda_3$. Since $(Z_{00}, Z_{pm})_{p\geq 1, 1\leq m\leq 2^p}$ are standard normal variables, we have

$$\begin{aligned}\prod_{(p,m)\in\Lambda_3} P_{pm}(\varepsilon) &\geq \prod_{(p,m)\in\Lambda_3, p\geq 1} \mathbb{P}\Big(Z_{pm} \in \Big[-\frac{\delta}{2c_{pm}(\alpha)\sqrt{\varepsilon}}, \frac{\delta}{2c_{pm}(\alpha)\sqrt{\varepsilon}}\Big]\Big)\\ &= \prod_{(p,m)\in\Lambda_3, p\geq 1} \Big(1 - \sqrt{\frac{2}{\pi}}\int_{\delta/(2c_{pm}(\alpha)\sqrt{\varepsilon})}^{\infty} e^{-u^2/2}du\Big).\end{aligned}$$

Now according to (13) and our choice of $\alpha$, $c_{pm}(\alpha)\leq 1, \lim_{p\to\infty} c_{pm}(\alpha) = 0$. Therefore, for $\epsilon > 0$ such that $\epsilon < \delta^2$ and all $p \geq 1, 1\leq m \leq 2^p$ we may estimate (see proof of Theorem 3.1)

$$\int_{\delta/(2c_{pm}(\alpha)\sqrt{\varepsilon})}^{\infty} \exp(-\frac{x^2}{2})dx \leq \exp(-\frac{\delta^2}{8c_{pm}(\alpha)^2\epsilon}).$$

In order to prove that $\prod_{(p,m)\in\Lambda_3} P_{pm}(\varepsilon)$ converges to 1 as $\epsilon\to 0$, by the elementary inequality $-x\leq \ln(1-x)$ for $x\in ]0,1[$ it suffices to prove that $\sum_{p\geq 1, 1\leq m\leq 2^p} \exp(-\frac{\delta^2}{8c_{pm}(\alpha)^2\varepsilon})$ converges to 0 as $\epsilon\to 0$. This is in fact the case due to (13). We deduce

$$\lim_{\varepsilon\to 0} \prod_{(p,m)\in\Lambda_3} P_{pm}(\varepsilon) = 1. \tag{21}$$

3. Next, we estimate the contribution of $\Lambda_4$. Indeed, $|\Lambda_4| < \infty$ and by definition $[\xi_{pm} - \delta, \xi_{pm}+\delta]$ contains a small neighborhood of the origin for any $(p,m)\in\Lambda_4$. We obtain

$$\lim_{\varepsilon\to 0} \prod_{(p,m)\in\Lambda_4} P_{pm}(\varepsilon) = 1. \tag{22}$$

4. Since $|\Lambda_2| < \infty$, its definition immediately gives

$$\lim_{\varepsilon\to 0} \prod_{(p,m)\in\Lambda_2} P_{pm}(\varepsilon) = 2^{-|\Lambda_2|}. \tag{23}$$

5. Let us finally estimate the contribution of $\Lambda_1$. We define

$$\bar{\xi}_{pm} = \begin{cases} \xi_{pm} - \delta, & \text{if } \xi_{pm} > \delta,\\ -(\xi_{pm}+\delta), & \text{if } \xi_{pm} < -\delta.\end{cases}$$

Since for $(p,m)\in\Lambda_1$ $Z_{pm}$ has a standard normal law, Theorem 3.1 implies

$$\lim_{\epsilon\to 0} \epsilon \ln P_{pm}(\varepsilon) = -\frac{\bar{\xi}_{pm}^2}{2c_{pm}(\alpha)^2}.$$

Since $|\Lambda_1| < \infty$, we therefore have

$$\lim_{\epsilon\to 0} \epsilon \ln \prod_{[p,m)\in\Lambda_1} P_{pm}(\varepsilon) = -\sum_{(p,m)\in\Lambda_1} \frac{\bar{\xi}_{pm}^2}{2c_{pm}(\alpha)^2}. \tag{24}$$

6. Using (21), (22), (23) and (24), we can deduce (17) if we are able to compare

$$\sum_{(p,m)\in\Lambda_1} \frac{\bar{\xi}_{pm}^2}{2c_{pm}(\alpha)^2} \qquad \text{with} \qquad \inf_{f\in B_\delta^\alpha(\psi)} I(f).$$

By Theorem 1.4 any function $f \in \mathcal{C}_0^\alpha \cap \mathcal{H}_1$ has the Schauder representation

$$f = \eta_{00}G_{00} + \sum_{p\geq 1, 1\leq m\leq 2^p} \frac{\eta_{pm}}{c_{pm}(\alpha)}\, G_{pm}, \quad \text{with } (\eta_{00}, (\eta_{pm})_{p\geq 0, 1\leq m\leq 2^p}) \in l^\infty(\mathbb{R}).$$

The derivative satisfies $\dot{f} = \eta_{00}H_{00} + \sum_{p\geq 1,1\leq m\leq 2^p} \frac{\eta_{pm}}{c_{pm}(\alpha)}\, H_{pm}$, and since $(H_{00}, (H_{pm})_{p\geq 1, 1\leq m\leq 2^p})$, is an orthonormal system in $L^2([0,1])$, we obtain

$$\frac{1}{2}\int_0^1 \dot{f}(s)^2\, ds = \sum_{p\geq 0, 1\leq m\leq 2^p} \frac{\eta_{pm}^2}{2c_{pm}(\alpha)^2}.$$

So the statement of the Theorem is an immediate consequence of the equality

$$\begin{aligned}\inf_{f\in B_\delta^\alpha(\psi)\cap\mathcal{H}_1} \frac{1}{2}\int_0^1 \dot{f}(s)^2\, ds \quad &= \quad \inf\Big\{ \sum_{p\geq 0, 0\leq m\leq 2^p} \frac{\eta_{pm}^2}{2c_{pm}(\alpha)^2},\ \text{with } \eta_{pm} \in ]\xi_{pm}-\delta, \xi_{pm}+\delta[ \Big\} \\ &= \sum_{(p,m)\in\Lambda_1} \frac{\bar{\xi}_{pm}^2}{2c_{pm}(\alpha)^2}.\end{aligned}$$

□

# 4 Paradifferential calculus and Young integration

In this section we develop the basic tools that will be required for our rough path integral in terms of Schauder functions. We shall formally decompose the integral into three components. In these terms, we shall derive Young's integral.

Before we continue, let us slightly change notation. We want to get rid of the factor $2^{-p/2}$ in (1), and therefore we define for $p \geq 0$ and $0 \leq m \leq 2^p$ the rescaled functions

$$\chi_{pm} := 2^{\frac{p}{2}} H_{pm} \qquad \text{and} \qquad \varphi_{pm} := 2^{\frac{p}{2}} G_{pm},$$

as well as $\varphi_{-10} := G_{-10} \equiv 1$. Then we have for $p \in \mathbb{N}$ and $1 \leq m \leq 2^p$

$$\|\varphi_{pm}(t)\|_\infty = \varphi_{pm}(t_{pm}^1) = 2^{\frac{p}{2}} \int_{t_{pm}^0}^{t_{pm}^1} 2^{\frac{p}{2}} \mathrm{d}s = 2^p \left( \frac{2m-1}{2^{p+1}} - \frac{2m-2}{2^{p+1}} \right) = \frac{1}{2},$$

so that $\|\varphi_{pm}\|_\infty \leq 1$ for all $p, m$. The expansion of $f$ in terms of $(\varphi_{pm})$ is given by $f_k = \sum_{p=0}^k \sum_{m=0}^{2^p} f_{pm}\varphi_{pm}$, where $f_{-10} := f(1)$, and $f_{00} := f(1) - f(0)$ and for $p \in \mathbb{N}$ and $m \geq 1$

$$f_{pm} := 2^{-p}\langle \chi_{pm}, \mathrm{d}f\rangle = 2f\left(t_{pm}^1\right) - f\left(t_{pm}^0\right) - f\left(t_{pm}^2\right) = f_{t_{pm}^0, t_{pm}^1} - f_{t_{pm}^1, t_{pm}^2}.$$

We write $\langle \chi_{pm}, \mathrm{d}f\rangle := 2^p f_{pm}$ for all values of $(p, m)$, despite not having defined $\chi_{-10}$.

**Definition 4.1.** For $\alpha > 0$ and $f\colon [0,1] \to \mathbb{R}^d$ the norm $\|\cdot\|_\alpha$ is defined as

$$\|f\|_\alpha := \sup_{pm} 2^{p\alpha} |f_{pm}|,$$

and we write

$$\mathcal{C}^\alpha := \mathcal{C}^\alpha(\mathbb{R}^d) := \Big\{ f \in C([0,1], \mathbb{R}^d) : \|f\|_\alpha < \infty \Big\}.$$

According to Theorem 1.3, we may indeed use the same name for the sequence space norm as before for the function space norm. And the old space $\mathcal{C}^\alpha$ is identical with the one just defined. For $\alpha \in (0,1)$, we have $\mathcal{C}^\alpha = C^\alpha([0,1], \mathbb{R}^d)$.

**Littlewood-Paley notation.** We will employ notation inspired from Littlewood-Paley theory. For $p \geq -1$ and $f \in C([0,1])$ we define

$$\Delta_p f := \sum_{m=0}^{2^p} f_{pm} \varphi_{pm} \qquad \text{and} \qquad S_p f := \sum_{q \leq p} \Delta_q f.$$

We will occasionally refer to $(\Delta_p f)$ as the Schauder blocks of $f$. Note that

$$\mathcal{C}^\alpha = \{ f \in C([0,1], \mathbb{R}^d) : \|(2^{p\alpha} \|\Delta_p f\|_\infty)_p\|_{\ell^\infty} < \infty \}.$$

## 4.1 The paraproduct in terms of Schauder functions

Here we introduce a "paradifferential calculus" in terms of Schauder functions. Paradifferential calculus is usually formulated in terms of Littlewood-Paley blocks and was initiated by Bony [Bon81]. For a gentle introduction see [BCD11].

We will need to study the regularity of $\sum_{p,m} u_{pm} \varphi_{pm}$, where $u_{pm}$ are functions and not constant coefficients. For this purpose we define the following space of sequences of functions.

**Definition 4.2.** If $(u_{pm} : p \geq -1, 0 \leq m \leq 2^p)$ is a family of affine functions of the form $u_{pm} : [t^0_{pm}, t^2_{pm}] \to \mathbb{R}^d$, we set for $\alpha > 0$

$$\|(u_{pm})\|_{\mathcal{A}^\alpha} := \sup_{p,m} 2^{p\alpha} \|u_{pm}\|_\infty,$$

where it is understood that $\|u_{pm}\|_\infty := \max_{t \in [t^0_{pm}, t^2_{pm}]} |u_{pm}(t)|$. The space $\mathcal{A}^\alpha := \mathcal{A}^\alpha(\mathbb{R}^d)$ is then defined as

$$\mathcal{A}^\alpha := \Big\{ (u_{pm})_{p \geq -1, 0 \leq m \leq 2^p} : u_{pm} \in C([t^0_{pm}, t^2_{pm}], \mathbb{R}^d) \text{ is affine and } \|(u_{pm})\|_{\mathcal{A}^\alpha} < \infty \Big\}.$$

Before proving a regularity estimate for affine expansions, let us establish an auxiliary result.

**Lemma 4.3.** *Let $s < t$ and let $f : [s,t] \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$ and $g : [s,t] \to \mathbb{R}^d$ be affine functions. Then for all $r \in (s,t)$ and for all $h > 0$ with $r - h \in [s,t]$ and $r + h \in [s,t]$ we have*

$$|(fg)_{r-h,r} - (fg)_{r,r+h}| \leq 8|t-s|^{-2} h^2 \|f\|_\infty \|g\|_\infty. \tag{25}$$

*Proof.* For $f(r) = a_1 + (r-s)b_1$ and $g(r) = a_2 + (r-s)b_2$ we have

$$|(fg)_{r-h,r} - (fg)_{r,r+h}| = |2f(r)g(r) - f(r-h)g(r-h) - f(r+h)g(r+h)| = |-2h^2 b_1 b_2|.$$

Now $f_{s,t} = b_1(t-s)$ so that $|b_1| \leq 2|t-s|^{-1}\|f\|_\infty$, and similarly for $b_2$. $\square$

We now prove a regularity estimate for affine expansions.

**Lemma 4.4.** *Let $\alpha \in (0,2)$ and let $(u_{pm}) \in \mathcal{A}^\alpha$. Then $\sum_{p,m} u_{pm}\varphi_{pm} \in \mathcal{C}^\alpha$, and*

$$\Big\| \sum_{p,m} u_{pm}\varphi_{pm} \Big\|_\alpha \lesssim \|(u_{pm})\|_{\mathcal{A}^\alpha}.$$

*Proof.* We need to examine the coefficients $2^{-q}\langle \chi_{qn}, \mathrm{d}(\sum_{pm} u_{pm}\varphi_{pm})\rangle$. The cases $(q,n) = (-1,0)$ and $(q,n) = (0,0)$ are easy. So let $q \geq 0$ and $1 \leq n \leq 2^q$. If $p > q$, then $\varphi_{pm}(t^i_{qn}) = 0$ for $i = 0,1,2$ and for all $m$, and therefore

$$2^{-q}\Big\langle \chi_{qn}, \mathrm{d}\Big(\sum_{p,m} u_{pm}\varphi_{pm}\Big)\Big\rangle = 2^{-q}\sum_{p\leq q}\sum_{m} \langle \chi_{qn}, \mathrm{d}(u_{pm}\varphi_{pm})\rangle.$$

If $p < q$, there is at most one $m_0$ with $\langle \chi_{qn}, \mathrm{d}(u_{pm_0}\varphi_{pm_0})\rangle \neq 0$. The support of $\chi_{qn}$ is then contained in $[t^0_{pm_0}, t^1_{pm_0}]$ or in $[t^1_{pm_0}, t^2_{pm_0}]$ and $u_{pm_0}$ and $\varphi_{pm_0}$ are affine on these intervals. So (25) yields, with $|t-s| = 2^{-p}, h = 2^{-q}$

$$\begin{aligned}\sum_m |2^{-q}\langle \chi_{qn}, \mathrm{d}(u_{pm}\varphi_{pm})\rangle| &= \sum_m |(u_{pm}\varphi_{pm})_{t^0_{qn},t^1_{qn}} - (u_{pm}\varphi_{pm})_{t^1_{qn},t^2_{qn}}| \\ &\lesssim 2^{2p}2^{-2q}\|u_{pm}\|_\infty\|\varphi_{pm}\|_\infty \lesssim 2^{p(2-\alpha)-2q}\|(u_{pm})\|_{\mathcal{A}^\alpha}.\end{aligned}$$

For $p = q$ we have $\varphi_{qn}(t^0_{qn}) = \varphi_{qn}(t^2_{qn}) = 0$ and $\varphi_{qn}(t^1_{qn}) = 1/2$, and thus

$$\sum_m |2^{-q}\langle \chi_{qn}, \mathrm{d}(u_{qm}\varphi_{qm})\rangle| = \Big|(u_{qn}\varphi_{qn})_{t^0_{qn},t^1_{qn}} - (u_{qn}\varphi_{qn})_{t^1_{qn},t^2_{qn}}\Big| = |u(t^1_{qn})| \lesssim 2^{-\alpha q}\|(u_{pm})\|_{\mathcal{A}^\alpha}.$$

Combining these estimate and using that $\alpha < 2$, we obtain

$$2^{-q}\Big|\Big\langle \chi_{qn}, \mathrm{d}\Big(\sum_{pm} u_{pm}\varphi_{pm}\Big)\Big\rangle\Big| \lesssim \sum_{p\leq q} 2^{p(2-\alpha)-2q}\|(u_{pm})\|_{\mathcal{A}^\alpha} \simeq 2^{-\alpha q}\|(u_{pm})\|_{\mathcal{A}^\alpha},$$

which completes the proof. $\square$

The following paraproducts will be essential in the decomposition of the integrals we investigate.

**Lemma 4.5.** *Let $\beta \in (0,2)$, let $v \in C([0,1], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$, and $w \in \mathcal{C}^\beta(\mathbb{R}^d)$. Then*

$$\pi_<(v,w) := \sum_{p=0}^{\infty} S_{p-1}v\Delta_p w \in \mathcal{C}^\beta(\mathbb{R}^n) \quad \textit{and} \quad \|\pi_<(v,w)\|_\beta \lesssim \|v\|_\infty\|w\|_\beta. \tag{26}$$

*Proof.* We have $\pi_<(v,w) = \sum_{p,m} u_{pm}\varphi_{pm}$ with $u_{pm} = (S_{p-1}v)|_{[t^0_{pm},t^2_{pm}]}w_{pm}$. For every $(p,m)$, the function $(S_{p-1}v)|_{[t^0_{pm},t^2_{pm}]}$ is the linear interpolation of $v$ between $t^0_{pm}$ and $t^2_{pm}$. As $\|(S_{p-1}v)|_{[t^0_{pm},t^2_{pm}]}w_{pm}\|_\infty \leq 2^{-p\beta}\|v\|_\infty\|w\|_\beta$, setting $u_{pm} = S_{p-1}v|_{[t^0_{pm},t^2_{pm}]}w_{pm}$, we obtain

$$\|(u_{pm})\|_{\mathcal{A}_\alpha} \leq \|v\|_\infty\|w\|_\beta.$$

Hence the statement follows from Lemma 4.4. $\square$

## 4.2 A decomposition of Young's integral

In this section we construct Young's integral using Schauder expansions. If $v \in \mathcal{C}^\alpha$ and $w \in \mathcal{C}^\beta$, we formally define

$$\int_0^\cdot v(s)\mathrm{d}w(s) := \sum_{p,m}\sum_{q,n} v_{pm}w_{qn}\int_0^\cdot \varphi_{pm}(s)\mathrm{d}\varphi_{qn}(s) = \sum_{p,q}\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s).$$

We show that this definition makes sense provided $\alpha+\beta > 1$. We identify three components of the integral that behave quite differently. This will be our starting point towards an extension of the integral beyond the Young regime.

In a first step, we have to estimate the Schauder coefficients of the iterated integrals of Schauder functions arising in our double sum.

**Lemma 4.6.** *Let $p > q \geq 0$. Then*

$$\langle \varphi_{pm}, \chi_{qn}\rangle = 2^{-p-2}\chi_{qn}(t^0_{pm}), \text{ and } |\langle \varphi_{pm}, \chi_{qn}\rangle| = 2^{p+q-2(p\vee q)-2} \tag{27}$$

*for all $m, n$. If $p = q$, then $\langle \varphi_{pm}, \chi_{pn}\rangle = 0$, except if $p = q = 0$, in which case the integral is bounded by 1. If $0 \leq p < q$, then for all $(m,n)$ we have*

$$\langle \varphi_{pm}, \chi_{qn}\rangle = -2^{-q-2}\chi_{pm}\left(t^0_{qn}\right), \text{ and } |\langle \varphi_{pm}, \chi_{qn}\rangle| = 2^{p+q-2(p\vee q)-2}. \tag{28}$$

*If $p = -1$, then the integral is bounded by 1.*

*Proof.* The cases $p = q$ and $p = -1$ are easy, so let $p > q \geq 0$. Since $\chi_{qn} \equiv \chi_{qn}(t^0_{pm})$ on the support of $\varphi_{pm}$, we have

$$\int_0^1 \varphi_{pm}(s)\mathrm{d}\varphi_{qn}(s) = \chi_{qn}(t^0_{pm})\int_0^1 \varphi_{pm}(s)\mathrm{d}s = \chi_{qn}(t^0_{pm})2^{-p-2}.$$

If $0 \leq p < q$, then integration by parts and (27) imply (28). □

Next we estimate the coefficients of iterated integrals in the Schauder basis.

**Lemma 4.7.** *Let $i, p \geq -1$, $q \geq 0$, $0 \leq j \leq 2^i$, $0 \leq m \leq 2^p$, $0 \leq n \leq 2^q$. Then*

$$2^{-i}\Big|\Big\langle \chi_{ij}, \mathrm{d}\Big(\int_0^\cdot \varphi_{pm}\chi_{qn}\mathrm{d}s\Big)\Big\rangle\Big| \leq 2^{-2(i\vee p\vee q)+p+q}, \tag{29}$$

*except if $p < q = i$. In this case we only have the worse estimate*

$$2^{-i}\Big|\Big\langle \chi_{ij}, \mathrm{d}\Big(\int_0^\cdot \varphi_{pm}\chi_{qn}\mathrm{d}s\Big)\Big\rangle\Big| \leq 1. \tag{30}$$

*Proof.* We have $\langle \chi_{-10}, \mathrm{d}(\int_0^\cdot \varphi_{pm}\chi_{qn}\mathrm{d}s)\rangle = 0$ for all $(p,m)$ and $(q,n)$. So let $i \geq 0$. If $i < p\vee q$, then $\chi_{ij}$ is constant on the support of $\varphi_{pm}\chi_{qn}$, and therefore Lemma 4.6 gives

$$2^{-i}\,|\langle \chi_{ij}, \varphi_{pm}\chi_{qn}\rangle| \leq |\langle \varphi_{pm}, \chi_{qn}\rangle| \leq 2^{p+q-2(p\vee q)} = 2^{-2(i\vee p\vee q)+p+q}.$$

Now let $i > q$. Then $\chi_{qn}$ is constant on the support of $\chi_{ij}$, and therefore another application of Lemma 4.6 implies that

$$2^{-i}\,|\langle \chi_{ij}, \varphi_{pm}\chi_{qn}\rangle| = 2^{q-i}|\langle \varphi_{pm}, \chi_{ij}\rangle| \leq 2^{q-i}2^{p+i-2(p\vee i)} = 2^{-2(i\vee p\vee q)+p+q}.$$

The only remaining case is $i = q \geq p$, in which

$$2^{-i} |\langle \chi_{ij}, \varphi_{pm}\chi_{qn}\rangle| \leq 2^i \int_{t_{ij}^0}^{t_{ij}^2} \varphi_{pm}(s)\mathrm{d}s \leq \|\varphi_{pm}\|_\infty \leq 1.$$

□

**Corollary 4.8.** *Let $i, p \geq -1$ and $q \geq 0$. Let $v \in C([0,1], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$ and $w \in C([0,1], \mathbb{R}^d)$. Then*

$$\Big\|\Delta_i\Big(\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s)\Big)\Big\|_\infty \lesssim 2^{-(i\vee p\vee q)-i+p+q}\|\Delta_p v\|_\infty\|\Delta_q w\|_\infty, \tag{31}$$

*except if $i = q > p$. In this case we only have the worse estimate*

$$\Big\|\Delta_i\Big(\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s)\Big)\Big\|_\infty \lesssim \|\Delta_p v\|_\infty\|\Delta_q w\|_\infty. \tag{32}$$

*Proof.* The case $i = -1$ is easy, so let $i \geq 0$. We have

$$\Delta_i\Big(\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s)\Big) = \sum_{j,m,n} v_{pm}w_{qn}\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle\varphi_{ij}.$$

For fixed $j$, there are at most $2^{(i\vee p\vee q)-i}$ non-vanishing terms in the double sum. Hence, we obtain from Lemma 4.7 that

$$\begin{aligned}\Big\|\sum_{m,n} v_{pm}w_{qn}\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle\varphi_{ij}\Big\|_\infty &\lesssim 2^{(i\vee p\vee q)-i}\|\Delta_p v\|_\infty\|\Delta_q w\|_\infty(2^{-2(i\vee p\vee q)+p+q} + \mathbf{1}_{i=q>p}) \\ &= (2^{-(i\vee p\vee q)-i+p+q} + \mathbf{1}_{i=q>p})\|\Delta_p v\|_\infty\|\Delta_q w\|_\infty.\end{aligned}$$

□

**Corollary 4.9.** *Let $i, p, q \geq -1$. Let $v \in C([0,1], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$ and $w \in C([0,1], \mathbb{R}^d)$. Then for $p \vee q \leq i$ we have*

$$\|\Delta_i(\Delta_p v\Delta_q w)\|_\infty \lesssim 2^{-(i\vee p\vee q)-i+p+q}\|\Delta_p v\|_\infty\|\Delta_q w\|_\infty, \tag{33}$$

*except if $i = q > p$ or $i = p > q$, in which case we only have the worse estimate*

$$\|\Delta_i(\Delta_p v\Delta_q w)\|_\infty \lesssim \|\Delta_p v\|_\infty\|\Delta_q w\|_\infty. \tag{34}$$

*If $p > i$ or $q > i$, then $\Delta_i(\Delta_p v\Delta_q w) \equiv 0$.*

*Proof.* The case $p = -1$ or $q = -1$ is easy. Otherwise we apply integration by parts and note that the estimates (31) and (32) are symmetric in $p$ and $q$. If for example $p > i$, then $\Delta_p(v)(t_{ij}^k) = 0$ for all $k, j$, which implies that $\Delta_i(\Delta_p v\Delta_q w) = 0$. □

We now come to a natural formal decomposition of our integral $\int_0^\cdot v(s)\mathrm{d}w(s)$ into three terms. They all have individual regularity properties to be derived from the estimates (31) and (32). More precisely, (32) indicates that the series $\sum_{p<q}\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s)$ is rougher than the remainder $\sum_{p\geq q}\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s)$. Integration by parts gives

$$\sum_{p<q}\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s) = \pi_<(v,w) - \sum_{p<q}\sum_{m,n} v_{pm}w_{qn}\int_0^\cdot \varphi_{qn}(s)\mathrm{d}\varphi_{pm}(s).$$

This motivates us to decompose the integral into three components, namely

$$\sum_{p,q}\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s) = L(v,w) + S(v,w) + \pi_<(v,w).$$

Here $L$ is defined as the antisymmetric *Lévy area*. It can be shown that $L$ is closely related to the Lévy area of certain dyadic martingales. We set:

$$\begin{aligned} L(v,w) &:= \sum_{p>q}\sum_{m,n}(v_{pm}w_{qn} - v_{qn}w_{pm})\int_0^\cdot \varphi_{pm}\mathrm{d}\varphi_{qn} \\ &= \sum_p \left(\int_0^\cdot \Delta_p v \mathrm{d}S_{p-1}w - \int_0^\cdot \mathrm{d}(S_{p-1}v)\Delta_p w\right). \end{aligned}$$

The *symmetric part* $S$ is defined as

$$\begin{aligned} S(v,w) &:= \sum_{m,n\leq 1} v_{0m}w_{0n}\int_0^\cdot \varphi_{0m}\mathrm{d}\varphi_{0n} + \sum_{p\geq 1}\sum_m v_{pm}w_{pm}\int_0^\cdot \varphi_{pm}\mathrm{d}\varphi_{pm} \\ &= \sum_{m,n\leq 1} v_{0m}w_{0n}\int_0^\cdot \varphi_{0m}\mathrm{d}\varphi_{0n} + \frac{1}{2}\sum_{p\geq 1}\Delta_p v\Delta_p w, \end{aligned}$$

and $\pi_<$ is the paraproduct defined in (26). As we observed in Lemma 4.5, $\pi_<(v,w)$ is always well defined, and it inherits the regularity of $w$. Let us study $S$ and $L$.

**Lemma 4.10.** *Let $\alpha,\beta\in(0,1)$ be such that $\alpha+\beta>1$. Then $L$ is a bounded bilinear operator from $\mathcal{C}^\alpha\times\mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta}$.*

*Proof.* We only argue for $\sum_p\int_0^\cdot \Delta_p v\mathrm{d}S_{p-1}w$ The term $-\int_0^\cdot \mathrm{d}(S_{p-1}v)\Delta_p w$ can be treated with the same arguments. Corollary 4.8 (more precisely (31)) implies that for $i\geq 0$

$$\begin{aligned} \Big\|\Delta_i\Big(\sum_p\int_0^\cdot \Delta_p v\mathrm{d}S_{p-1}w\Big)\Big\|_\infty &= \Big\|\sum_p \Delta_i\Big(\int_0^\cdot \Delta_p v\mathrm{d}S_{p-1}w\Big)\Big\|_\infty \\ &\leq \sum_{p\leq i}\sum_{q<p}\Big\|\Delta_i\Big(\int_0^\cdot \Delta_p v\mathrm{d}\Delta_q w\Big)\Big\|_\infty + \sum_{p>i}\sum_{q<p}\Big\|\Delta_i\Big(\int_0^\cdot \Delta_p v\mathrm{d}\Delta_q w\Big)\Big\|_\infty \\ &\leq \Big(\sum_{p\leq i}\sum_{q<p}2^{-2i+p+q}2^{-p\alpha}\|v\|_\alpha 2^{-q\beta}\|w\|_\beta + \sum_{p>i}\sum_{q<p}2^{-i+q}2^{-p\alpha}\|v\|_\alpha 2^{-q\beta}\|w\|_\beta\Big) \\ &\lesssim_{\alpha+\beta} 2^{-i(\alpha+\beta)}\|v\|_\alpha\|w\|_\beta, \end{aligned}$$

where we used $1-\alpha>0$ and $1-\beta>0$. For the second series we also used that $\alpha+\beta>1$. $\square$

Unlike the Lévy area $L$, the symmetric part $S$ is always well defined. It is also smooth.

**Lemma 4.11.** *Let $\alpha, \beta \in (0,1)$. Then $S$ is a bounded bilinear operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta}$.*

*Proof.* This is shown using the same arguments as in the proof of Lemma 4.10. □

In conclusion, the integral consists of three components. The Lévy area $L(v, w)$ is only defined if $\alpha + \beta > 1$, but then it is smooth. The symmetric part $S(v, w)$ is always defined and smooth. And the paraproduct $\pi_<(v, w)$ is always defined, but it is rougher than the other components. To summarize:

**Theorem 4.12** (Young's integral)**.** *Let $\alpha, \beta \in (0,1)$ be such that $\alpha + \beta > 1$, and let $v \in \mathcal{C}^\alpha$ and $w \in \mathcal{C}^\beta$. Then the integral*

$$I(v, \mathrm{d}w) := \sum_{p,q} \int_0^\cdot \Delta_p v \mathrm{d}\Delta_q w = L(v,w) + S(v,w) + \pi_<(v,w) \in \mathcal{C}^\beta$$

*satisfies $\|I(v, \mathrm{d}w)\|_\beta \lesssim \|v\|_\alpha \|w\|_\beta$ and*

$$\|I(v, \mathrm{d}w) - \pi_<(v,w)\|_{\alpha+\beta} \lesssim \|v\|_\alpha \|w\|_\beta. \tag{35}$$

# 5 Paracontrolled paths, pathwise integration beyond Young

In this section we construct a rough path integral in terms of Schauder functions. Let us first motivate by an example (see [IP15]) what might be missing for two functions that serve as integrand and integrator in a rough integral, in case for the Hölder coefficients we have the inequality $\alpha + \beta \le 1$. Since we use trigonometric functions instead of Haar and Schauder functions we shall briefly switch the domain from $[0,1]$ to $[-1,1]$.

*Example* 5.1. Let us consider for $m \in \mathbb{N}$ the functions $(f^m, g^m)\colon [-1,1] \to \mathbb{R}^2$ with components given by

$$f^m_t := \sum_{k=1}^m a_k \sin(2^k \pi t) \quad \text{and} \qquad g^m_t := \sum_{k=1}^m a_k \cos(2^k \pi t), \quad t \in [-1,1],$$

where $a_k := 2^{-\alpha k}$ and $\alpha \in [0,1]$. Set $f := \lim_{m\to\infty} f^m$, $g := \lim_{m\to\infty} g^m$. These functions are $\alpha$-Hölder continuous uniformly in $m$. Indeed, let $s, t \in [-1,1]$ and choose $k \in \mathbb{N}$ such that $2^{-k-1} \le |s-t| \le 2^{-k}$. Then we can estimate as follows

$$\begin{aligned}
|f^m_t - f^m_s| &= \left| \sum_{l=1}^m a_l 2 \cos(2^{l-1}\pi(s+t)) \sin(2^{l-1}\pi(s-t)) \right| \\
&\le 2 \sum_{l=1}^k |a_l| |\sin(2^{l-1}\pi(s-t))| + 2 \sum_{l=k+1}^\infty |a_l| \le 2 \sum_{l=1}^k |a_l| 2^{l-1}\pi|s-t| + 2 \sum_{l=k+1}^\infty |a_l| \\
&\le \sum_{l=1}^k 2^{l-\alpha l}\pi|s-t| + 2^{-\alpha(k+1)+1} \frac{1}{1-2^{-\alpha}} \le \frac{2^{(k+1)(1-\alpha)}-1}{2^{1-\alpha}-1}\pi|s-t| + \frac{2^{1-\alpha}}{1-2^{-\alpha}}|s-t|^\alpha \\
&\le \frac{2^{(k+1)(1-\alpha)}-1}{2^{1-\alpha}-1}\pi 2^{-k(1-\alpha)}|s-t|^\alpha + \frac{2^{1-\alpha}}{1-2^{-\alpha}}|s-t|^\alpha \le C|s-t|^\alpha
\end{aligned}$$

for some constant $C > 0$ independent of $m \in \mathbb{N}$. Analogously, we can get the $\alpha$-Hölder continuity of $g^m$. Furthermore, it can be seen with the same estimate that $(f^m)$ converges uniformly to $f$, $(g^m)$ to $g$ and thus also in $\alpha$-Hölder topology. The limit functions $f$ and $g$ are not $\beta$-Hölder continuous for every $\beta > \alpha$. In order to see this, choose $s = 0$ and $t = t_n = 2^{-n}$ for $n \in \mathbb{N}$ and observe that

$$\frac{|f_{t_n} - f_0|}{|t_n - 0|^\beta} = \sum_{k=1}^{n-1} 2^{-\alpha k + \beta n} \sin(2^{k-n}\pi) \geq 2^{(\beta-\alpha)n+\alpha},$$

which obviously tends to infinity with $n$.

Let us now show that $(f, g)$ possesses no Lévy area. For this purpose, fix $\alpha \in [0, 1]$ and $m \in \mathbb{N}$. Then Lévy's area for $(f^m, g^m)$ is given by

$$\begin{aligned}
&\int_{-1}^1 f_s^m \mathrm{d}g_s^m - \int_{-1}^1 g_s^m \mathrm{d}f_s^m \\
&= -\sum_{k,l=1}^m a_k a_l \int_{-1}^1 \big(\sin(2^k\pi s)\sin(2^l\pi s)2^l\pi + \cos(2^l\pi s)\cos(2^k\pi s)2^k\pi\big)\mathrm{d}s \\
&= -\sum_{k,l=1}^m a_k a_l \big(2^l\pi \int_{-1}^1 \frac{1}{2}(\cos((2^k - 2^l)\pi s) - \cos((2^k + 2^l)\pi s))\mathrm{d}s \\
&\quad + 2^k\pi \int_{-1}^1 (\cos((2^k - 2^l)\pi s) + \cos((2^k + 2^l)\pi s))\mathrm{d}s\big) \\
&= -2\sum_{k=1}^m a_k^2 2^k \pi = -2\sum_{k=1}^m 2^{(1-2\alpha)k}\pi.
\end{aligned}$$

This quantity diverges as $m$ tends to infinity for $\alpha < \frac{1}{2}$. Since $(f^m, g^m)$ converges to $(f, g)$ in the $\alpha$-Hölder topology, we can use this result to choose partition sequences of $[-1, 1]$ along which Riemann sums approximating the Lévy area of $(f, g)$ diverge as well. This shows that $(f, g)$ possesses no Lévy area.

We will now see that $f$ possesses no *fractional Taylor expansion* up to first order with respect to $g$ and vice versa. We will name this expansion a *control relationship* between $f$ and $g$. So the example will show that neither $f$ is controlled by $g$ nor vice versa. For this purpose, note that for $-1 \leq s \leq t \leq 1$, and $0 \neq f_s^g \in \mathbb{R}$

$$\begin{aligned}
|f_{s,t} - f_s^g g_{s,t}| &= \Big|\sum_{k=1}^\infty a_k[(\sin(2^k\pi t) - \sin(2^k\pi s)) - f_s^g(\cos(2^k\pi t) - \cos(2^k\pi s))]\Big| \\
&= \Big|2\sum_{k=1}^\infty a_k\big[\sin(2^{k-1}\pi(s-t))\cos(2^{k-1}\pi(s+t)) + f_s^g \sin(2^{k-1}\pi(s+t))\sin(2^{k-1}\pi(s-t))\big]\Big| \\
&= \Big|2\sum_{k=1}^\infty a_k \sin(2^{k-1}\pi(s-t))\sqrt{1 + (f_s^g)^2}\sin(2^{k-1}\pi(s+t) + \arctan((f_s^g)^{-1}))\Big|.
\end{aligned}$$

Let us now investigate Hölder regularity at $s = 0$. First, assume $f_0^g > 0$, and take $t = 2^{-n}$ to obtain

$$\begin{aligned}
\frac{|f_{0,2^n} - f_0^g g_{0,2^n}|}{2^{-\beta n}} &= 2^{\beta n}\Big|2\sum_{k=1}^n a_k \sin(2^{k-1-n}\pi)\sqrt{1 + (f_0^g)^2}\sin(2^{k-1-n}\pi + \arctan((f_0^g)^{-1}))\Big| \\
&\geq 2^{(\beta-\alpha)n}\sin\big(\frac{\pi}{2} + \arctan((f_0^g)^{-1})\big).
\end{aligned}$$

For $f_0^g < 0$ the same estimates work for $t_n = -2^{-n}$ instead. Therefore, the Hölder regularity at 0 cannot be better than $\alpha$ and in particular $f$ cannot be controlled by $g$ for $\frac{1}{2} > \alpha$. Switching the roles of $f$ and $g$ with similar arguments leads to the same conclusion.

## 5.1 Paracontrolled paths

For going beyond the Young limit in the theory of integration of rough paths, according to the example just given the concept of control will play the essential role. In fact, we will assume that integrand and integrator are controlled by a joint rough function for which we know that the three terms obtained in the decomposition given in Section 4.2 make sense. The notion of control of rough functions generalizes the approximation by linear or quadratic terms in Taylor's formula of differential calculus into the domain of fractional approximation orders. We use control by paraproducts, and obtain the following notion of paracontrolled paths.

**Definition 5.2.** Let $\alpha > 0$ and $v \in \mathcal{C}^\alpha(\mathbb{R}^d)$. We define

$$\mathcal{D}_v^\alpha := \mathcal{D}_v^\alpha(\mathbb{R}^n) := \left\{(f, f^v) \in \mathcal{C}^\alpha(\mathbb{R}^n) \times \mathcal{C}^\alpha(\mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)) : f^\sharp = f - \pi_<(f^v, v) \in \mathcal{C}^{2\alpha}(\mathbb{R}^n)\right\}.$$

If $(f, f^v) \in \mathcal{D}_v^\alpha$, then $f$ is called *paracontrolled* by $v$. The function $f^v$ is called the *derivative* of $f$ with respect to $v$. Abusing notation, we write $f \in \mathcal{D}_v^\alpha$ if it is clear from the context what the derivative $f^v$ is supposed to be. We equip $\mathcal{D}_v^\alpha$ with the norm

$$\|f\|_{v,\alpha} := \|f^v\|_\alpha + \|f^\sharp\|_{2\alpha}.$$

If $v \in \mathcal{C}^\alpha$ and $(\tilde{f}, \tilde{f}^{\tilde{v}}) \in \mathcal{D}_{\tilde{v}}^\alpha$, then we also write

$$d_{\mathcal{D}^\alpha}(f, \tilde{f}) := \|f^v - \tilde{f}^{\tilde{v}}\|_\alpha + \|f^\sharp - \tilde{f}^\sharp\|_{2\alpha}.$$

*Example* 5.3. Let $\alpha + \beta > 1$ and $v \in \mathcal{C}^\alpha$, $w \in \mathcal{C}^\beta$. Then by (35), the Young integral $I(v, \mathrm{d}w)$ is in $\mathcal{D}_w^\alpha$, with derivative $v$.

*Example* 5.4. If $2\alpha < 1$ and $v \in \mathcal{C}^\alpha$, then $(f, f^v) \in \mathcal{D}_v^\alpha$ if and only if $|f_{s,t} - f_s^v v_{s,t}| \lesssim |t-s|^{2\alpha}$ and in that case

$$\|f^v\|_\infty + \sup_{s \neq t} \frac{|f_{s,t}^v|}{|t-s|^\alpha} + \sup_{s \neq t} \frac{|f_{s,t} - f_s^v v_{s,t}|}{|t-s|^{2\alpha}} \lesssim \|f\|_{v,\alpha}(1 + \|v\|_\alpha).$$

Indeed we have $|f_s^v v_{s,t} - \pi_<(f^v, v)_{s,t}| \lesssim |t-s|^{2\alpha}\|f^v\|_\alpha \|v\|_\alpha$, which can be shown using similar arguments as for Lemma B.2 in [GIP12]. In other words, for $\alpha \in (0, 1/2)$ the space $\mathcal{D}_v^\alpha$ coincides with a space of *controlled paths.*

## 5.2 A basic commutator estimate

Here we prove the commutator estimate which will be the main ingredient in the construction of the integral $I(f, \mathrm{d}g)$, where $f$ is paracontrolled by $v$ and $g$ is paracontrolled by $v$, and where we assume that the Lévy area of the control $L(v, v)$ exists.

**Propostion 5.5.** *Let $\alpha \in (0,1)$, and assume that $2\alpha < 1 < 3\alpha$. Let $f, v, w \in \mathcal{C}^\alpha$. Then the "commutator"*

$$
\begin{aligned}
C(f,v,w) &:= L(\pi_<(f,v),w) - I(f, \mathrm{d}L(v,w)) \qquad (36)\\
&:= \lim_{N\to\infty} [L(S_N(\pi_<(f,v)), S_N w) - I(f, \mathrm{d}L(S_N v, S_N w))]\\
&= \lim_{N\to\infty} \sum_{p\le N}\sum_{q<p} \Bigg[ \int_0^\cdot \Delta_p(\pi_<(f,v))(s)\mathrm{d}\Delta_q w(s) - \int_0^\cdot f(s)\Delta_p v(s)\mathrm{d}\Delta_q w(s)\\
&\qquad - \Big( \int_0^\cdot \mathrm{d}(\Delta_q(\pi_<(f,v)))(s)\Delta_p w(s) - \int_0^\cdot f(s)\mathrm{d}(\Delta_q v)(s)\Delta_p w(s)\Big)\Bigg]
\end{aligned}
$$

*converges in $\mathcal{C}^{3\alpha-\varepsilon}$ for all $\varepsilon > 0$. Moreover,*

$$\|C(f,v,w)\|_{3\alpha} \lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha.$$

*Proof.* We only argue for the first difference in (36), i.e. for

$$X_N := \sum_{p\le N}\sum_{q<p} \left[ \int_0^\cdot \Delta_p(\pi_<(f,v))(s)\mathrm{d}\Delta_q w(s) - \int_0^\cdot f(s)\Delta_p v(s)\mathrm{d}\Delta_q w(s)\right]. \qquad (37)$$

The second difference can be handled using the same arguments. First we prove that $(X_N)$ converges uniformly. Then we show that $\|X_N\|_{3\alpha}$ stays uniformly bounded. This will imply the desired result, since bounded sets in $\mathcal{C}^{3\alpha}$ are relatively compact in $\mathcal{C}^{3\alpha-\varepsilon}$.

To prove uniform convergence, note that

$$
\begin{aligned}
X_N - X_{N-1} &= \sum_{q<N} \left[ \int_0^\cdot \Delta_N(\pi_<(f,v))(s)\mathrm{d}\Delta_q w(s) - \int_0^\cdot f(s)\Delta_N v(s)\mathrm{d}\Delta_q w(s)\right]\\
&= \sum_{q<N} \Bigg[ \sum_{j\le N}\sum_{i<j} \int_0^\cdot \Delta_N(\Delta_i f \Delta_j v)(s)\mathrm{d}\Delta_q w(s)\\
&\qquad - \sum_{j\ge N}\sum_{i\le j} \int_0^\cdot \Delta_j(\Delta_i f\Delta_N v)(s)\mathrm{d}\Delta_q w(s)\Bigg], \qquad (38)
\end{aligned}
$$

where for the second term it is possible to take the infinite sum over $j$ outside of the integral because $\sum_j \Delta_j g$ converges uniformly to $g$ and because $\Delta_q w$ is a finite variation path. We also used that $\Delta_N(\Delta_i f \Delta_j v) = 0$ whenever $i > N$ or $j > N$. The terms in (38) for $j = N$ cancel. These cancellations are crucial, since they eliminate terms for which we only have the worse estimate (34) in Corollary 4.9. We obtain

$$
\begin{aligned}
X_N - X_{N-1} &= \sum_{q<N}\sum_{j<N}\sum_{i<j} \int_0^\cdot \Delta_N(\Delta_i f\Delta_j v)(s)\mathrm{d}\Delta_q w(s) - \sum_{q<N}\int_0^\cdot \Delta_N(\Delta_N f\Delta_N v)(s)\mathrm{d}\Delta_q w(s)\\
&\quad - \sum_{q<N}\sum_{j>N}\sum_{i<j} \int_0^\cdot \Delta_j(\Delta_i f\Delta_N v)(s)\mathrm{d}\Delta_q w(s)\\
&\quad - \sum_{q<N}\sum_{j>N} \int_0^\cdot \Delta_j(\Delta_j f\Delta_N v)(s)\mathrm{d}\Delta_q w(s). \qquad (39)
\end{aligned}
$$

Note that $\|\partial_t \Delta_q w\|_\infty \lesssim 2^q \|\Delta_q w\|_\infty$. Hence, an application of Corollary 4.9, where we use (33) for the first three terms and (34) for the fourth term, yields

$$\begin{aligned}
\|X_N - X_{N-1}\|_\infty \lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha \Bigg[ & \sum_{q<N} \sum_{j<N} \sum_{i<j} 2^{-2N+i+j} 2^{-i\alpha} 2^{-j\alpha} 2^{q(1-\alpha)} \\
& + \sum_{q<N} 2^{-2N\alpha} 2^{q(1-\alpha)} + \sum_{q<N} \sum_{j>N} \sum_{i<j} 2^{-2j+i+N} 2^{-i\alpha} 2^{-N\alpha} 2^{q(1-\alpha)} \\
& + \sum_{q<N} \sum_{j>N} 2^{-j\alpha} 2^{-N\alpha} 2^{q(1-\alpha)} \Bigg] \\
\lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha 2^{-N(3\alpha-1)}, &
\end{aligned} \tag{40}$$

where in the last step we used $3\alpha > 1$. This gives us the uniform convergence of $(X_N)$.

Next let us show that $\|X_N\|_{3\alpha} \lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha$ for all $N$. Similarly to (39) we obtain for $n \in \mathbb{N}$

$$\begin{aligned}
\Delta_n X_N = \sum_{p \le N} \sum_{q<p} \Delta_n \Bigg[ & \sum_{j<p} \sum_{i<j} \int_0^\cdot \Delta_p(\Delta_i f \Delta_j v)(s) \mathrm{d}\Delta_q w(s) - \int_0^\cdot \Delta_p(\Delta_p f \Delta_p v)(s) \mathrm{d}\Delta_q w(s) \\
& - \sum_{j>p} \sum_{i \le j} \int_0^\cdot \Delta_j(\Delta_i f \Delta_p v)(s) \mathrm{d}\Delta_q w(s) \Bigg],
\end{aligned}$$

and therefore by Corollary 4.8

$$\begin{aligned}
\|\Delta_n X_N\|_\infty \lesssim \sum_p \sum_{q<p} \Bigg[ & \sum_{j<p} \sum_{i<j} 2^{-(n\vee p)-n+p+q} \|\Delta_p(\Delta_i f \Delta_j v)\|_\infty \|\Delta_q w\|_\infty \\
& + 2^{-(n\vee p)-n+p+q} \|\Delta_p(\Delta_p f \Delta_p v)\|_\infty \|\Delta_q w\|_\infty \\
& + \sum_{j>p} \sum_{i \le j} 2^{-(n\vee j)-n+j+q} \|\Delta_j(\Delta_i f \Delta_p v)\|_\infty \|\Delta_q w\|_\infty \Bigg].
\end{aligned}$$

Now we apply Corollary 4.9, where for the last term we distinguish the cases $i < j$ and $i = j$. Using that $1 - \alpha > 0$, we get

$$\begin{aligned}
\|\Delta_n X_N\|_\infty \lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha \sum_p 2^{p(1-\alpha)} \Bigg[ & \sum_{j<p} \sum_{i<j} 2^{-(n\vee p)-n+p} 2^{-2p} 2^{i(1-\alpha)} 2^{j(1-\alpha)} \\
& + 2^{-(n\vee p)-n+p} 2^{-p\alpha} 2^{-p\alpha} \\
& + \sum_{j>p} \sum_{i<j} 2^{-(n\vee j)-n+j} 2^{-2j+i(1-\alpha)+p(1-\alpha)} \\
& + \sum_{j>p} 2^{-(n\vee j)-n+j} 2^{-j\alpha-p\alpha} \Bigg] \\
\lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha 2^{-n(3\alpha)}, &
\end{aligned}$$

where we used both that $3\alpha > 1$ and that $2\alpha < 1$. □

*Remark* 5.6. If $2\alpha = 1$, we can apply Proposition 5.5 with $\alpha - \varepsilon$ to obtain that $C(f, v, w) \in \mathcal{C}^{3\alpha-\varepsilon}$ for every sufficiently small $\varepsilon > 0$. If $2\alpha > 1$, then we are in the Young setting and there is no need to introduce the commutator.

### 5.3 Pathwise integration for paracontrolled paths

In this section we apply the commutator estimate to construct the rough path integral under the assumption that the Lévy area exists for a given reference path.

**Theorem 5.7.** *Let $\alpha \in (1/3, 1)$, and assume that $2\alpha \neq 1$. Let $v \in \mathcal{C}^\alpha(\mathbb{R}^d)$ and assume that the Lévy area*

$$L(v, v) := \lim_{N\to\infty} \big(L(S_N v^k, S_N v^\ell)\big)_{1\le k\le d, 1\le \ell\le d}$$

*converges uniformly and that* $\sup_N \|L(S_N v, S_N v)\|_{2\alpha} < \infty$*. Let $f \in \mathcal{D}_v^\alpha(\mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$. Then $I(S_N f, \mathrm{d}S_N v)$ converges in $\mathcal{C}^{\alpha-\varepsilon}$ for all $\varepsilon > 0$. Denoting the limit by $I(f, \mathrm{d}v)$, we have*

$$\|I(f, \mathrm{d}v)\|_\alpha \lesssim \|f\|_{v,\alpha}\big(\|v\|_\alpha + \|v\|_\alpha^2 + \|L(v, v)\|_{2\alpha}\big).$$

*Moreover, $I(f, \mathrm{d}v) \in \mathcal{D}_v^\alpha$ with derivative $f$ and*

$$\|I(f, \mathrm{d}v)\|_{v,\alpha} \lesssim \|f\|_{v,\alpha}\big(1 + \|v\|_\alpha^2 + \|L(v, v)\|_{2\alpha}\big).$$

*Proof.* If $2\alpha > 1$, everything follows from the Young case, Theorem 4.12, so let $2\alpha < 1$. We decompose

$$\begin{aligned} I(S_N f, \mathrm{d}S_N v) = {}& S(S_N f, S_N v) + \pi_<(S_N f, S_N v) + L(S_N f^\sharp, S_N v) \\ & + [L(S_N \pi_<(f^v, v), S_N v) - I(f^v, \mathrm{d}L(S_N v, S_N v))] + I(f^v, \mathrm{d}L(S_N v, S_N v)). \end{aligned}$$

Convergence then follows from Proposition 5.5 and Theorem 4.12. The limit is given by

$$I(f, \mathrm{d}v) = S(f, v) + \pi_<(f, v) + L(f^\sharp, v) + C(f^v, v, v) + I(f^v, \mathrm{d}L(v, v)),$$

from where we easily deduce the claimed bounds. □

## 6 Construction of the Lévy area for hypercontractive processes

To apply our theory, it remains to construct the Lévy area for suitable stochastic processes. We discuss the example of hypercontractive stochastic processes whose covariance function satisfies a certain "finite variation" property.

Let $X\colon [0, 1] \to \mathbb{R}^d$ be a centered continuous stochastic process, such that $X^i$ is independent of $X^j$ for $i \neq j$. We write $R$ for its covariance function, $R\colon [0, 1]^2 \to \mathbb{R}^{d\times d}$ and $R(s, t) := (E(X^i_s X^j_t))_{1\le i,j\le d}$. The increment of $R$ over a rectangle $[s, t] \times [u, v] \subseteq [0, 1]^2$ is defined as

$$R_{[s,t]\times[u,v]} := R(t, v) + R(s, u) - R(s, v) - R(t, u) := (E(X^i_{s,t} X^j_{u,v}))_{1\le i,j\le d}.$$

Let us make the following two assumptions.

($\rho$–var) There exists $C > 0$ such that for all $0 \le s < t \le 1$ and for every partition $s = t_0 < t_1 < \cdots < t_n = t$ of $[s, t]$ we have

$$\sum_{i,j=1}^n |R_{[t_{i-1},t_i]\times[t_{j-1},t_j]}|^\rho \le C|t - s|.$$

(HC) The process $X$ is hypercontractive, i.e. for every $m, n \in \mathbb{N}$ and every $r \geq 1$ there exists $C_{r,m,n} > 0$ such that for every polynomial $P : \mathbb{R}^n \to \mathbb{R}$ of degree $m$, for all $i_1, \dots, i_n \in \{1, \dots, d\}$, and for all $t_1, \dots, t_n \in [0, 1]$

$$E(|P(X^{i_1}_{t_1}, \dots, X^{i_n}_{t_n})|^{2r}) \leq C_{r,m,n} E(|P(X^{i_1}_{t_1}, \dots, X^{i_n}_{t_n})|^2)^r.$$

These conditions are taken from [FV10a], where under even more general assumptions it is shown that it is possible to construct the iterated integrals $I(X, \mathrm{d}X)$, and that $I(X, \mathrm{d}X)$ is the limit of $(I(X^n, \mathrm{d}X^n))_{n\in\mathbb{N}}$ under a wide range of smooth approximations $(X^n)_n$ that converge to $X$.

*Example* 6.1. Condition (HC) is satisfied by all Gaussian processes. More generally, it is satisfied by every process "living in a fixed Gaussian chaos"; see [Jan97], Theorem 3.50. Slightly oversimplifying things, this is the case if $X$ is given by polynomials of fixed degree and iterated integrals of fixed order with respect to a Gaussian reference process.

Prototypical examples of processes living in a fixed chaos are Hermite processes. They are defined for $H \in (1/2, 1)$ and $k \in \mathbb{N}$, $k \geq 1$ as

$$Z^{k,H}_t = C(H, k) \int_{\mathbb{R}^k} \left( \int_0^t \prod_{i=1}^k (s - y_i)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} \mathrm{d}s \right) \mathrm{d}B_{y_1} \dots \mathrm{d}B_{y_k},$$

where $(B_y)_{y\in\mathbb{R}}$ is a standard Brownian motion, and $C(H, k)$ is a normalization constant. In particular, $Z^{k,H}$ lives in the Wiener chaos of order $k$. The covariance of $Z^{k,H}$ is

$$E(Z^{k,H}_s Z^{k,H}_t) = \frac{1}{2} \left( t^{2H} + s^{2H} + |t - s|^{2H} \right)$$

Since $Z^{1,H}$ is Gaussian, it is just the fractional Brownian motion with Hurst parameter $H$. For $k = 2$ we obtain the Rosenblatt process. For further details about Hermite processes see [PT11]. However, we should point out that it follows from Kolmogorov's continuity criterion that $Z^{k,H}$ is $\alpha$–Hölder continuous for every $\alpha < H$. Since $H \in (1/2, 1)$, Hermite processes are amenable to Young integration, and it is trivial to construct $L(Z^{k,H}, Z^{k,H})$.

*Example* 6.2. Condition ($\rho$–var) is satisfied by Brownian motion with $\rho = 1$. More generally it is satisfied by the fractional Brownian motion with Hurst index $H$, for which $\rho = 1/(2H)$. It is also satisfied by the fractional Brownian bridge with Hurst index $H$. A general criterion that implies condition ($\rho$–var) is the one of Coutin and Qian [CQ02]: If $E(|X^i_{s,t}|^2) \lesssim |t - s|^{2H}$ and $|E(X^i_{s,s+h} X^i_{t,t+h})| \lesssim |t - s|^{2H-2} h^2$ for $i = 1, \dots, d$, then ($\rho$–var) is satisfied for $\rho = 1/(2H)$. For details and further examples see [FV10b], Section 15.2.

**Lemma 6.3.** *Assume that the stochastic process $X : [0, 1] \to \mathbb{R}$ satisfies ($\rho$–var). Then we have for all $p \geq -1$ and for all $M, N \in \mathbb{N}$ with $M \leq N \leq 2^p$ that*

$$\sum_{m_1, m_2 = M}^{N} |E(X_{pm_1} X_{pm_2})|^\rho \lesssim (N - M + 1) 2^{-p}. \tag{41}$$

*Proof.* The case $p \leq 0$ is easy so let $p \geq 1$. It suffices to note that

$$\begin{aligned} E(X_{pm_1} X_{pm_2}) &= E\left( (X_{t^0_{pm_1}, t^1_{pm_1}} - X_{t^1_{pm_1}, t^2_{pm_1}})(X_{t^0_{pm_2}, t^1_{pm_2}} - X_{t^1_{pm_2}, t^2_{pm_2}}) \right) \\ &= \sum_{i_1, i_2 = 0,1} (-1)^{i_1 + i_2} R_{[t^{i_1}_{pm_1}, t^{i_1+1}_{pm_1}] \times [t^{i_2}_{pm_2}, t^{i_2+1}_{pm_2}]}, \end{aligned}$$

and that $\{t^i_{pm} : i = 0, 1, 2, m = M, \dots, N\}$ partitions the interval $[(M - 1)2^{-p}, N2^{-p}]$. □

**Lemma 6.4.** *Let $X, Y : [0,1] \to \mathbb{R}$ be independent, centered, continuous processes, both satisfying ($\rho$–var) for some $\rho \in [1,2]$. Then for all $i, p \geq -1$, $q < p$, and $0 \leq j \leq 2^i$*

$$E\Big[\Big|\sum_{m\leq 2^p}\sum_{n\leq 2^q} X_{pm}Y_{qn}\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle\Big|^2\Big] \lesssim 2^{(p\vee i)(1/\rho-4)}2^{(q\vee i)(1-1/\rho)}2^{-i}2^{p(4-3/\rho)}2^{q/\rho}.$$

*Proof.* Since $p > q$, for every $m$ there exists exactly one $n(m)$, such that $\varphi_{pm}\chi_{qn(m)}$ is not identically zero. Hence, we can apply the independence of $X$ and $Y$ to obtain

$$\begin{aligned} &E\Big[\Big|\sum_{m\leq 2^p}\sum_{n\leq 2^q} X_{pm}Y_{qn}\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle\Big|^2\Big] \\ &\quad\leq \sum_{m_1,m_2=0}^{2^p} \big|E(X_{pm_1}X_{pm_2})E(Y_{qn(m_1)}Y_{qn(m_2)})\langle 2^{-i}\chi_{ij}, \varphi_{pm_1}\chi_{qn(m_1)}\rangle\langle 2^{-i}\chi_{ij}, \varphi_{pm_2}\chi_{qn(m_2)}\rangle\big|. \end{aligned}$$

Let us write $M_j := \{m : 0 \leq m \leq 2^p, \langle \chi_{ij}, \varphi_{pm}\chi_{qn(m)}\rangle \neq 0\}$. We also write $\rho'$ for the conjugate exponent of $\rho$, i.e. $1/\rho + 1/\rho' = 1$. Hölder's inequality and Lemma 4.7 imply

$$\begin{aligned} &\sum_{m_1,m_2\in M_j} \big|E(X_{pm_1}X_{pm_2})E(Y_{qn(m_1)}Y_{qn(m_2)})\langle 2^{-i}\chi_{ij}, \varphi_{pm_1}\chi_{qn(m_1)}\rangle\langle 2^{-i}\chi_{ij}, \varphi_{pm_2}\chi_{qn(m_2)}\rangle\big| \\ &\quad\lesssim \Big(\sum_{m_1,m_2\in M_j}\big|E(X_{pm_1}X_{pm_2})\big|^\rho\Big)^{1/\rho}\Big(\sum_{m_1,m_2\in M_j}\big|E(Y_{qn(m_1)}Y_{qn(m_2)})\big|^{\rho'}\Big)^{1/\rho'}(2^{-2(p\vee i)+p+q})^2. \end{aligned}$$

Now write $N_j$ for the set of $n$ for which $\chi_{ij}\chi_{qn}$ is not identically zero. For every $\bar{n} \in N_j$ there are $2^{p-q}$ numbers $m \in M_j$ with $n(m) = \bar{n}$. Hence

$$\begin{aligned} &\Big(\sum_{m_1,m_2\in M_j}\big|E(Y_{qn(m_1)}Y_{qn(m_2)})\big|^{\rho'}\Big)^{1/\rho'} \\ &\qquad\lesssim (2^{2(p-q)})^{1/\rho'}\Big(\Big(\max_{n_1,n_2\in N_j}\big|E(Y_{qn_1}Y_{qn_2})\big|\Big)^{\rho'-\rho}\sum_{n_1,n_2\in N_j}\big|E(Y_{qn_1}Y_{qn_2})\big|^\rho\Big)^{1/\rho'}, \end{aligned}$$

where we used that $\rho \in [1,2]$ and therefore $\rho' - \rho \geq 0$ (for $\rho' = \infty$ we interpret the right hand side as $\max_{n_1,n_2\in N_j}|E(Y_{qn_1}Y_{qn_2})|$). Lemma 6.3 implies that $\big(\big|E(Y_{qn_1}Y_{qn_2})\big|^{\rho'-\rho}\big)^{1/\rho'} \lesssim 2^{-q(1/\rho-1/\rho')}$. Similarly we apply Lemma 6.3 to the sum over $n_1, n_2$, and we obtain

$$\begin{aligned} &(2^{2(p-q)})^{1/\rho'}\Big(\Big(\max_{n_1,n_2\in N_j}\big|E(Y_{qn_1}Y_{qn_2})\big|\Big)^{\rho'-\rho}\sum_{n_1,n_2\in N_j}\big|E(Y_{qn_1}Y_{qn_2})\big|^\rho\Big)^{1/\rho'} \\ &\qquad\lesssim (2^{2(p-q)})^{1/\rho'}2^{-q(1/\rho-1/\rho')}(|N_j|2^{-q})^{1/\rho'} = 2^{(q\vee i)/\rho'}2^{-i/\rho'}2^{2p/\rho'}2^{q(-2/\rho'-1/\rho)} \\ &\qquad= 2^{(q\vee i)(1-1/\rho)}2^{i(1/\rho-1)}2^{2p(1-1/\rho)}2^{q(1/\rho-2)}, \end{aligned}$$

where we used that $|N_j| = 2^{(q\vee i)-i}$. Since $|M_j| = 2^{(p\vee i)-i}$, another application of Lemma 6.3 yields

$$\Big(\sum_{m_1,m_2\in M_j}\big|E(X_{pm_1}X_{pm_2})\big|^\rho\Big)^{1/\rho} \lesssim 2^{(p\vee i)/\rho}2^{-i/\rho}2^{-p/\rho}.$$

The result now follows by combining these estimates:

$$
\begin{aligned}
E\Big[\Big|&\sum_{m\le 2^p}\sum_{n\le 2^q} X_{pm}Y_{qn}\langle 2^{-i}\chi_{ij},\varphi_{pm}\chi_{qn}\rangle\Big|^2\Big]\\
&\lesssim \Big(\sum_{m_1,m_2\in M_j}\big|E(X_{pm_1}X_{pm_2})\big|^{\rho}\Big)^{1/\rho}\Big(\sum_{m_1,m_2\in M_j}\big|E(Y_{qn(m_1)}Y_{qn(m_2)})\big|^{\rho'}\Big)^{1/\rho'}(2^{-2(p\vee i)+p+q})^2\\
&\lesssim \big(2^{(p\vee i)/\rho}2^{-i/\rho}2^{-p/\rho}\big)\big(2^{(q\vee i)(1-1/\rho)}2^{i(1/\rho-1)}2^{2p(1-1/\rho)}2^{q(1/\rho-2)}\big)\big(2^{-4(p\vee i)+2p+2q}\big)\\
&= 2^{(p\vee i)(1/\rho-4)}2^{(q\vee i)(1-1/\rho)}2^{-i}2^{p(4-3/\rho)}2^{q/\rho}.
\end{aligned}
$$

□

**Theorem 6.5.** *Let $X\colon[0,1]\to\mathbb{R}^d$ be a continuous, centered stochastic process with independent components, and assume that $X$ satisfies (HC) and ($\rho$–var) for some $\rho\in[1,2)$. Then for every $\alpha\in(0,1/\rho)$ almost surely*

$$
\sum_{N\ge 0}\|L(S_NX,S_NX)-L(S_{N-1}X,S_{N-1}X)\|_\alpha<\infty,
$$

*and therefore $L(X,X)=\lim_{N\to\infty}L(S_NX,S_NX)$ is almost surely $\alpha$–Hölder continuous.*

*Proof.* First note that $L$ is antisymmetric, and in particular the diagonal of the matrix $L(S_NX,S_NX)$ is constantly zero. For $k,\ell\in\{1,\dots,d\}$ with $k\ne\ell$ we have

$$
\begin{aligned}
\|L(S_NX^k,&S_NX^\ell)-L(S_{N-1}X^k,S_{N-1}X^\ell)\|_\alpha\\
&=\Big\|\sum_{q<N}\sum_{m,n}(X^k_{Nm}X^\ell_{qn}-X^k_{qn}X^\ell_{Nm})\int_0^{\cdot}\varphi_{Nm}(s)\mathrm{d}\varphi_{qn}(s)\Big\|_\alpha\\
&\le\sum_{q<N}\Big\|\sum_{m,n}X^k_{Nm}X^\ell_{qn}\int_0^{\cdot}\varphi_{Nm}(s)\mathrm{d}\varphi_{qn}(s)\Big\|_\alpha+\sum_{q<N}\Big\|\sum_{m,n}X^\ell_{Nm}X^k_{qn}\int_0^{\cdot}\varphi_{Nm}(s)\mathrm{d}\varphi_{qn}(s)\Big\|_\alpha
\end{aligned}
$$

Let us argue for the first term on the right hand side, the arguments for the second one being identical. Let $r\ge 1$. Using the hypercontractivity condition (HC), we obtain

$$
\begin{aligned}
\sum_{i,N}\sum_{j\le 2^i}\sum_{q<N}&P\Big(\Big|\sum_{m,n}X^\ell_{Nm}X^k_{qn}\langle 2^{-i}\chi_{ij},\varphi_{Nm}\chi_{qn}\rangle\Big|>2^{-i\alpha}2^{-N/(2r)}2^{-q/(2r)}\Big)\\
&\le\sum_{i,N}\sum_{j\le 2^i}\sum_{q<N}E\Big(\Big|\sum_{m,n}X^\ell_{Nm}X^k_{qn}\langle 2^{-i}\chi_{ij},\varphi_{Nm}\chi_{qn}\rangle\Big|^{2r}\Big)2^{i\alpha 2r}2^{N+q}\\
&\lesssim\sum_{i,N}\sum_{j\le 2^i}\sum_{q<N}E\Big(\Big|\sum_{m,n}X^\ell_{Nm}X^k_{qn}\langle 2^{-i}\chi_{ij},\varphi_{Nm}\chi_{qn}\rangle\Big|^{2}\Big)^r2^{i\alpha 2r}2^{N+q}.
\end{aligned}
$$

Now we can apply Lemma 6.4 to bound this expression by

$$\sum_{i,N}\sum_{j\leq 2^i}\sum_{q<N}\left(2^{(N\vee i)(1/\rho-4)}2^{(q\vee i)(1-1/\rho)}2^{-i}2^{N(4-3/\rho)}2^{q/\rho}\right)^r 2^{i\alpha 2r}2^{N+q}$$

$$\begin{aligned}
&\lesssim \sum_i 2^i \sum_{N\leq i}\sum_{q<N} 2^{ir(2\alpha-4)}2^{Nr(4-3/\rho+1/r)}2^{qr(1/\rho+1/r)}\\
&\quad+\sum_i 2^i \sum_{N>i}\sum_{q\leq i} 2^{ir(2\alpha-1/\rho)}2^{Nr(1/r-2/\rho)}2^{qr(1/\rho+1/r)}\\
&\quad+\sum_i 2^i \sum_{N>i}\sum_{i<q<N} 2^{ir(2\alpha-1)}2^{Nr(1/r-2/\rho)}2^{qr(1+1/r)}\\
&\lesssim \sum_i 2^{ir(2\alpha+3/r-2/\rho)}+\sum_i\sum_{N>i}2^{ir(2\alpha+2/r)}2^{Nr(1/r-2/\rho)}\\
&\quad+\sum_i\sum_{N>i}2^{ir(2\alpha+1/r-1)}2^{Nr(1+2/r-2/\rho)}.
\end{aligned}$$

For $r\geq 1$ we have $1/r-2/\rho<0$, because $\rho<2$. Therefore, the sum over $N$ in the second term on the right hand side converges. If now we choose $r>1$ large enough so that $1+3/r-2/\rho<0$ (and then also $2\alpha+3/r-2/\rho<0$), then all three series on the right hand side are finite. Hence, Borel-Cantelli implies the existence of $C(\omega)>0$, such that for almost all $\omega\in\Omega$ and for all $N,i,j$ and $q<N$

$$\Big|\sum_{m,n}X^\ell_{Nm}(\omega)X^k_{qn}(\omega)\langle 2^{-i}\chi_{ij},\varphi_{Nm}\chi_{qn}\rangle\Big|\leq C(\omega)2^{-i\alpha}2^{-N/(2r)}2^{-q/(2r)}.$$

From here it is straightforward to see that for these $\omega$ we have

$$\sum_{N=0}^{\infty}\|L(S_NX(\omega),S_NX(\omega))-L(S_{N-1}X(\omega),S_{N-1}X(\omega))\|_\alpha<\infty.$$

□

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