Quantum optics

Maxwell equations

1.1 $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ ← Gauss law

1.2 $\vec{\nabla}\cdot\vec{B} = 0$ <- Gauss law for magnetism

1.3 $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ← Maxwell-Faraday law

Lenz law

1.4 $\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\left(\frac{\vec{J}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t}\right)$ ← Maxwell-Ampère law

It is possible to combine $\vec{E}$ and $\vec{B}$ into a (covariant, antysimmetric) tensor

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$F_{\alpha\beta}$ → electromagnetic tensor

Minkowski metric

[as usual $F^{\mu\nu} = \eta^{\mu\alpha} F_{\alpha\beta}\, \eta^{\beta\nu}$ $\quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$]

Planck, photoel effect, QM. De Broglie, QED '29 Dirac
Schrö, Heis
'47 Feyn.
Glaub

Quantum (noun)
quantum (adj)

G.I Taylor experiment. 1909 Glauber PR 84 '51

Add Haroche experiment.
QED

with the definition of $F^{\mu\nu}$, we can write Maxwell equations as

$$2.1 \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad 1.2 \ \& \ 1.3$$

$$B = \nabla \times \bar{A}$$
$$E = -\nabla \phi - \partial_t \bar{A}$$

$$2.2 \quad \partial_\mu F^{\mu\nu} = \frac{j^\nu}{c^2 \varepsilon_0} \qquad 1.1 \ \& \ 1.4$$

$$\partial_\mu = \left(\frac{1}{c} \partial_t , \partial_x \right)$$

with

$$A_\alpha = \left(\frac{\phi}{c} , -\vec{A} \right) \leftarrow \text{4-potential}$$

Scalar pot.

vector potential

and

$$j_\alpha = \left(c\rho , \vec{j} \right) \leftarrow \text{4-current}$$

Furthermore eqns. 2.1 & 2.2 can be combined into a single equation

$$③ \quad \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{j^\nu}{c^2 \varepsilon_0}$$

The price to pay for going from a field $F_{\mu\nu}$ to a potential representation (A_μ) is that A_μ is not uniquely determined:

$$A'_\mu = A_\mu + \partial_\mu \lambda$$

would have been an equally valid solution of (3). We could exploit this freedom by imposing an extra condition

(4) $\partial_\mu A^\mu = 0$ (Lorenz condition)

which allows us to write Maxwell's eqns. as

(5) $\Box A^\mu = \dfrac{j^\mu}{c^2 \varepsilon_0}$

$\partial^\nu \partial_\nu = \dfrac{1}{c^2} \dfrac{\partial^2}{\partial t^2} - \nabla^2 \rightarrow$ d'Alembert operator.

The Lorenz condition (4) is not sufficient to uniquely specify A^μ (we could have chosen any λ for which $\Box \lambda = 0$).

For our purposes it will be ok to focus on a situation in which charges are absent ($j^\mu = 0$) and where we give up the manifestly covariant form of Maxwell's equations. To fix the residual freedom allowed by the Lorenz gauge, we choose $A_0 = 0$, which translates to $\vec{\nabla} \cdot \vec{A} = 0$ (i.e. Coulomb gauge).

In empty space Maxwell's equations (5) can be written as

$$(6) \qquad \Box A^\mu = 0 \qquad\qquad \left(\frac{1}{c^2} \partial_t^2 - \nabla^2 \right) A^\mu = 0$$

which represent a wave equation for each component, for which we seek a solution of the form

$$(7) \qquad A^\mu = a \exp\left[-i k^\mu x_\mu \right] e^\mu(p) \qquad\qquad k^\mu = \left(\frac{\omega}{c}, \vec{k} \right)$$
$$x^\mu = (t, \vec{x})$$

substituting (7) into (6) we obtain

$$k^\mu k_\mu = 0 \qquad \Rightarrow \qquad \omega = c|\vec{k}|$$

moreover, the Coulomb condition entails:

$$e^0 \quad , \quad \bar{e} \cdot \bar{k} = 0$$

Implying that $\bar{e}$ is perpendicular to the direction of propagation

$\Rightarrow$ transversality of the e.m. waves

a. $\quad \varepsilon^1 = (0,1,0,0) \quad ; \quad \varepsilon^2 = (0,0,1,0)$ $\leftarrow$ linear polarisation

b. $\quad \varepsilon^1 = \frac{1}{\sqrt{2}} (0,1,i,0) \quad ; \quad \varepsilon^2 = \frac{1}{\sqrt{2}} (0,1,-i,0)$ $\leftarrow$ circular polarisation

We have shown that, in the Coulomb gauge $\vec{A} = (A_1, A_2, A_3)$ satisfies the equation

$$\Box A_i = 0 \qquad i = 1,2,3 \qquad (A^0 = 0 \text{ Coulomb gauge})$$

We separate the vector potential into two complex terms

$$\bar{A}(\bar{r},t) = \bar{A}^+(\bar{r},t) + \bar{A}^-(\bar{r},t)$$

$\bar{A}^+$ contains amplitudes which vary as

$$\exp[-i\omega t] \quad (\omega > 0)$$

$\bar{A}^-$ contains amplitudes which vary as $\exp[i\omega t]$ and $\bar{A}^- = (\bar{A}^+)^*$.

Restricting the field to a finite volume of space ($V \to \infty$ in the end) in order to work with a discrete set of modes, we can write

$$\bar{A}^+(\bar{r},t) = \sum a_k u_k(\bar{r}) \exp[-i\omega_k t]$$

The mode functions $u_k(\bar{r})$ satisfy the equation

$$\left(\bar{\nabla}^2 + \frac{\omega_k^2}{c^2}\right) u_k = 0$$

and, again, from the Coulomb gauge

$$\bar{\nabla}\,\bar{u}_k = 0 \quad \text{(transversality condition)}$$

The form of $\bar{u}_k$ depends on the choice of the physical volume under consideration. If we choose a cubical volume of side L with p.b.c. we have

$$u_k(\bar{r}) = \frac{1}{\sqrt{V}} \hat{e}_\lambda \exp\left[i \bar{k} \cdot \bar{r}\right] \qquad V = L^3 \qquad \lambda = 1,2 \text{ pol. idx}$$

$$\bar{k} = \frac{2\pi}{L}\left[n_x, n_y, n_z\right] \qquad n_i \in \mathbb{Z} \qquad \hat{e}_\lambda \perp \hat{k} \quad (\text{transv. again})$$

The vector potential can thus be written as

$$A(\bar{r},t) = \sum_k \left(\frac{\hbar}{2\omega_k \varepsilon_0}\right)^{\frac{1}{2}} \left\{ a_k u_k(\bar{r}) \exp\left[-i\omega_k t\right] + a_k^+ u_k^*(\bar{r}) \exp\left[i\omega_k t\right] \right\}$$

$$E = -\partial_t \bar{A}$$

$$E(\bar{r},t) = \sum_k i \left(\frac{\hbar \omega_k}{2\varepsilon_0}\right)^{\frac{1}{2}} \left\{ a_k u_k(\bar{r}) \exp\left[-i\omega_k t\right] - a_k^+ u_k^*(\bar{r}) \exp\left[i\omega_k t\right] \right\}$$

reminiscent of momentum.

$$\frac{\hat{P}^2}{2m} + \frac{1}{2} m \omega^2 \hat{X}^2 \qquad x = \frac{1}{\sqrt{2}}(a + a^+), \quad p = \frac{i}{\sqrt{2}}(a^+ - a)$$

Quantum E.M. $\rightarrow$ quantisation of a, a^+

a, a^+ bosons $\rightarrow$ $[a_k, a_{k'}] = [a^+_k, a^+_{k'}] = 0$, $[a_k, a^+_{k'}] = \delta_{kk'}$

why bosons?

The Hamiltonian of the e.m. field is given by

$$H_{EM} = \frac{1}{2}\varepsilon \int dV \left[|\bar{E}|^2 + c^2 |\bar{B}|^2 \right] \quad \leftarrow \text{ex}$$

its quantum form in terms of a^+_k, a_k is given by

$$H_{EM} = \sum_k \hbar\omega_k \left(a^+_k a_k + \frac{1}{2} \right) \quad \leftarrow \text{ex ?}$$

in addition, from the classical expression for the momentum of an E.M. field

$$\bar{P} = \frac{1}{\varepsilon_0} \int dV \, \bar{E} \times \bar{B}$$

we can conclude that

$$\bar{P} = \hbar \sum_k \bar{k}_k$$

and therefore the dispersion relation $\omega = c|\vec{k}|$ (taking into account a single mode) can be expressed in terms of energy and momentum of a photon as

$$E = c|\vec{p}|$$

The dynamics of the E.M. field is thus equivalent to the dynamics of independent harmonic oscillators for each mode. The excitations of the E.M. field can be represented in terms of number of photons (+ vacuum energy) <- discuss for each mode. The state of the system can be discussed in terms of a tensor product $|\psi\rangle = \bigotimes_k |\psi_k\rangle$, where $|\psi_k\rangle$ is the state vector associated with mode k.

Let's now consider 3 possible representations of the E.M. field (i.e. $|\psi\rangle$)

Fock states

The Hamiltonian H_{em} has eigenvalues $\hbar \omega_k n_k$

The corresponding eigenstates are written as $|n_k\rangle$ and are known as number or Fock states, they are eigenstates of the number operator

$$\hat{a}_k^\dagger a_k |n_k\rangle = n_k |n_k\rangle$$

moreover

$$\langle n_k | a_k^\dagger a_k | n_k\rangle = \| a_k |n_k\rangle \| \geq 0 \quad \Rightarrow \quad n_k \geq 0$$

Furthermore, the action of $\hat{a}_k^\dagger$ (raising operator) and $\hat{a}_k$ (lowering operator) is given by

1 $\hat{a}_k |n_k\rangle = \sqrt{n_k}\, |n_k - 1\rangle \qquad \hat{a}_k^\dagger |n_k\rangle = \sqrt{n_k + 1}\, |n_k + 1\rangle \Big| \Rightarrow$ SLOW

therefore $a_k, a_k^\dagger$ represent the creation ($\hat{a}_k^\dagger$) and annihilation of a photon with wavevector $\vec{k}$ and polarization $\hat{\varepsilon}_{\vec{k}}$

show this

and the relation $\hat{a}_k|n_k\rangle = \sqrt{n_k}\,|n_k-1\rangle$ implies that $n_k \in \mathbb{N}$ ($n_k = 0, 1 \dots \infty$).

We thus have that the vacuum state of a given mode (state with 0 photons) is defined by

$$\hat{a}_k|0\rangle = 0 .$$

A number state $|n_k\rangle$ can be obtained from $|0\rangle$ by successive applications of the creation operator

$$|n_k\rangle = \frac{(a^+_k)^{n_k}}{(n_k!)^{1/2}}|0\rangle$$

$|n_k\rangle$ are orthogonal and complete.

Fock states are a useful representation for states where the number of photons is very small (e.g. high-energy photons, cavity QED), they are not suitable when the number of photons is large. In this case fields are either a superposition (pure state) or a mixture (mixed states) of Fock states.

Coherent states

The introduction of field coherent states (C.S.) by Glauber [GL 1,2] was motivated, as we shall discuss later, by the necessity of factorising to all orders the E.M. field correlation function. In a sense (which can be made rigorous) C.S. are quantum states which can be employed for a (semi-) classical description of a quantum system, in our case the E.M. field.

Focusing on a single mode ($a \equiv a_k$), we have 3 equivalent definitions of field C.S. (f.c.s.)

1. eigenstates of $a \Rightarrow a|\alpha\rangle = \alpha|\alpha\rangle \quad \alpha \in \mathbb{C}$

2. $|\alpha\rangle$ can be obtained from $|0\rangle$ applying a displacement operator $D(\alpha) = \exp[\alpha a^+ - \alpha^* a]$

3. $|\alpha\rangle$ are states with a minimum uncertainty relationship, i.e:

$$(\Delta q)^2 (\Delta p)^2 = \frac{1}{4}$$

with $q = \frac{a^+ + a}{\sqrt{2}}$, $p = i\,\frac{a - a^+}{\sqrt{2}}$ and $(\Delta f)^2 = \langle\alpha|(f - \langle f\rangle)^2|\alpha\rangle$

N.B.: Def. 3 is not unique.

From def 1, we can obtain a representation of f.c.s. in terms of Fock states.

The general expansion of a c.s. on the Fock basis can be written as

$$|\alpha\rangle = \sum c_n |n\rangle$$

Since $a|\alpha\rangle = \alpha|\alpha\rangle$

$$a \sum c_n |n\rangle = \alpha \sum c_n |n\rangle ,$$

implying

$$\sum c_{n+1} \sqrt{n+1}\, |n\rangle = \sum \alpha c_n |n\rangle$$

leading to the following recursion relation

$$c_n = \frac{\alpha}{\sqrt{n}}\, c_{n-1}$$

and thus

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\, c_0$$

therefore

$$|\alpha\rangle = c_0 \sum \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle$$

Requiring $\langle\alpha|\alpha\rangle = 1$

$$|c_0|^2 \sum \frac{|\alpha|^{2n}}{n!} = |c_0|^2 \exp\left[|\alpha|^2\right] = 1$$

and thus

$$|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle$$

From this relation, it is possible to evaluate the probability distribution of photons (i.e. the probability distribution of having n photons) in a f.c.s

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n} \exp[-|\alpha|^2]}{n!} = \frac{\bar{n}^n \exp[-\bar{n}]}{n!}$$

which is a Poisson distribution: if, on average there are $\bar{n} = |\alpha|^2$ photons, what is the probability of having n of them?

N.B: for a Poisson distribution $\bar{n} = \langle (n-\bar{n})^2 \rangle$

Coherent states are not orthogonal, but

$$\langle \beta|\alpha\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) + \alpha\beta^*\right]$$

see A3 for the proof.

They are however complete (over-complete, since they're not orthogonal)

$$\mathbb{1} = \frac{1}{\pi}\int d^2\alpha \; |\alpha\rangle\langle\alpha|$$

(see A4 for the proof)

Squeezed states

As mentioned, the def 3 of f.c.s is not unique: the class of squeezed states is the one encompassing all minimum-uncertainty states (m.u.s.)

For a f.c.s., we have that

$$\langle \alpha | q | \alpha \rangle = \frac{1}{\sqrt{2}} (\alpha^* + \alpha)$$

$$\langle \alpha | q^2 | \alpha \rangle = \frac{1}{2} \langle \alpha | (a^\dagger + a)(a^\dagger + a) | \alpha \rangle = \frac{1}{2} (\alpha^{*2} + \alpha^2 + 2|\alpha|^2 + 1)$$

$$= \frac{1}{2} \left[(\alpha^* + \alpha)^2 + 1 \right]$$

Thus

$$(\Delta q)^2_\alpha = \left[\langle \alpha | q^2 | \alpha \rangle - \langle \alpha | q | \alpha \rangle^2 \right] = \frac{1}{2}$$

and similarly $(\Delta p)^2_\alpha = \frac{1}{2}$

Hence f.c.s are indeed m.u.s. with equal uncertainties in both quadratures (q, p). How can we generate states for which $\Delta q \neq \Delta p$?

$\Rightarrow$ Squeezing operator:

$$S(\zeta) = \exp\left[\zeta \frac{a^{\dagger 2}}{2} - \zeta^* \frac{a^2}{2} \right]$$

A squeezed state is defined by

$$|\xi,\alpha\rangle = D(\alpha)\, S(\xi)\, |0\rangle$$

The properties of a squeezed state can be illustrated by evaluating the expectation values of a, a^+, a^2, a^{+2} on $|\xi,\alpha\rangle$. To this end we calculate $S^+(\xi)\, D^+(\alpha)\, a\, D(\alpha)\, S(\xi)$.

Firstly, we show that

$$D^+(\alpha)\, a\, D(\alpha) = a + \alpha$$

From the definition of $D(\alpha)$ we have[1]

$$\exp[-\alpha a^+ + \alpha^* a]\, a\, \exp[\alpha a^+ - \alpha^* a] = a + [-\alpha a^+ + \alpha^* a, a] + \underbrace{\dots}_{=0}$$
$$+ (-\alpha)[a^+, a]$$
$$= a + \alpha$$

We can similarly evaluate $S^+(\xi)\, a\, S(\xi)$

$$\sum \frac{1}{n!}\left[-\frac{\xi}{2}(a^+)^2 + \frac{\xi^*}{2} a^2, a\right] = a + \left[-\frac{\xi}{2}(a^+)^2, a\right] + \xi\left[\frac{\xi^*}{2} a^2, a^+\right] + \frac{1}{3!}|\xi|^2\left[-\frac{\xi}{2}(a^+)^2, a\right] + \dots$$

$$a + \xi a^+ + \frac{1}{2}|\xi|^2 a + \frac{1}{3!}|\xi|^2 \xi\, a^+ + \dots = a \operatorname{ch}|\xi| + a^+ e^{i\varphi} \operatorname{sh}|\xi|$$

$\xi = |\xi| e^{i\varphi}$

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[1] Remembering that $\exp[B]\, A\, \exp[-B] = \sum_n \frac{1}{n!}[B, A]_n$

Therefore

$$\langle \alpha, \xi | a | \xi, \alpha \rangle = \langle 0 | S^{\dagger}(\xi)\, D^{\dagger}(\alpha)\, a\, D(\alpha)\, S(\xi) | 0 \rangle$$

$$= \langle 0 | \left(\alpha + c\, a + s\, a^{\dagger} \right) | 0 \rangle = \alpha$$

$$\begin{cases} c = \operatorname{ch} |\xi| \\ s = e^{i\varphi} \operatorname{th} |\xi| \end{cases}$$

Analogously

$$\langle \alpha, \xi | a^{\dagger} | \xi, \alpha \rangle = \alpha^{*}$$

and

$$\langle \alpha, \xi | a^{2} | \xi, \alpha \rangle = \langle 0 | (\alpha + c\, a + s\, a^{\dagger})^{2} | 0 \rangle = \alpha^{2} + c\, s$$

$$\langle \alpha, \xi | a^{\dagger 2} | \xi, \alpha \rangle = \langle 0 | (\alpha^{*} + c^{*} a^{\dagger} + s^{*} a)^{2} | 0 \rangle = \alpha^{*2} + c^{*} s^{*}$$

$$\langle \alpha, \xi | n | \xi, \alpha \rangle = \langle 0 | (\alpha + c^{*} a^{\dagger} + s^{*} a)(\alpha + c\, a + s\, a^{\dagger}) | 0 \rangle = |\alpha|^{2} + |s|^{2}$$

We can now calculate $(\Delta q)^{2}_{\xi\alpha}$ and $(\Delta p)^{2}_{\xi\alpha}$

$$(\Delta q)^{2}_{\xi\alpha} = \frac{1}{2} \langle \alpha, \xi | (a^{\dagger} + a)^{2} | \xi, \alpha \rangle - \frac{1}{2} \langle \alpha, \xi | (a^{\dagger} + a) | \xi, \alpha \rangle^{2}$$

$$= \frac{1}{2} \left[\langle \alpha, \xi | a^{\dagger 2} + a^{2} + 2n + 1 | \xi, \alpha \rangle - (\alpha^{*} + \alpha)^{2} \right]$$

$$= \frac{1}{2} \left[1 + 2|s|^{2} + c(s + s^{*}) \right] \overset{\varphi = 0}{=} \frac{1}{2} \left[1 + 2|s|^{2} + 2|s|c \right]$$

$$= \frac{1}{2} \left[|c|^{2} + |s|^{2} + 2|s|c \right]$$

Ex. 1

$$\langle \alpha, \xi | n | \xi, \alpha \rangle = \langle 0 | (\alpha + c^* a^\dagger + s^* a)(\alpha + c a + s a^\dagger) | 0 \rangle = |\alpha|^2 + |s|^2$$

$H =$

$$\Omega(|c|^2 + |s|^2) a^\dagger a + \Omega c s \, a^{\dagger 2} + \Omega c s^* a^2$$

$$\eta = cs = \frac{1}{2} e^{i\varphi} \Omega \, sh|\xi|$$

$$\Omega(|c|^2 + |s|^2) = \omega_0$$

$$\omega_0 = \Omega \, ch|\xi| \, , \quad \eta = \frac{1}{2} \Omega e^{i\varphi} sh|\xi|$$

$$\frac{2\eta}{\omega_0} = e^{i\varphi} \tanh|\xi|$$

Similarly

$$(\Delta p)^2_{\xi\alpha} = \frac{1}{2}\left[1 + 2|s|^2 - c(s + s^*)\right]$$

Therefore

$$(\Delta q)^2_{\xi\alpha}\,(\Delta p)^2_{\xi\alpha} = \frac{1}{4}\left[\left(1 + 2|s|^2\right)^2 - c^2\left(s + s^*\right)^2\right] = \frac{1}{4}\left[1 + \mathrm{sh}^2\, 2|\xi|\, \sin^2\varphi\right]$$

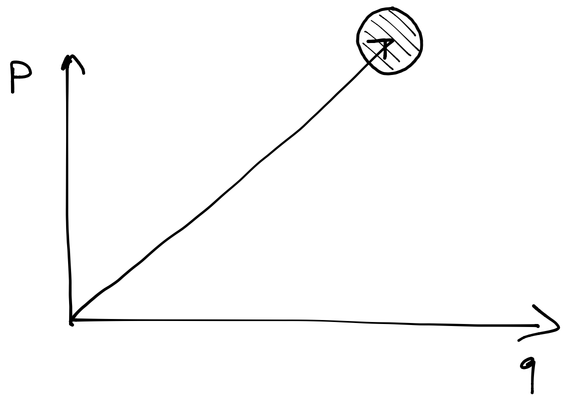
$$= \frac{1}{4} \quad \text{if} \quad \varphi = 0, \pi$$

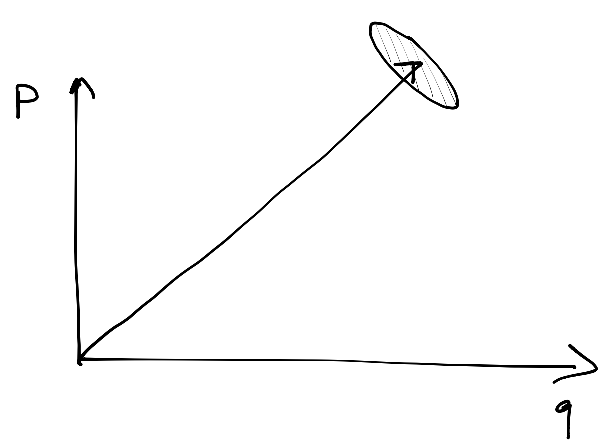
for $\varphi = 0$

$$(\Delta q)^2_{\xi\alpha} = \frac{1}{2}\left(\mathrm{ch}|\xi| + \mathrm{sh}|\xi|\right)^2 = \frac{1}{2}\exp\left[2\xi\right] > \frac{1}{2}$$

$$(\Delta p)^2_{\xi\alpha} = \frac{1}{2}\left(\mathrm{ch}|\xi| - \mathrm{sh}|\xi|\right)^2 = \frac{1}{2}\exp\left[-2\xi\right] < \frac{1}{2}$$

and for $\varphi = \pi$ we obtain the opposite squeezing effect





Appendix A

A.1

$$[A, B^n] = [A,B]\, n\, B^{n-1}$$

Proof

By induction:

$n=1$ $$[A,B] = [A,B]$$

$n \to n+1$ $$[A,B^{n+1}] = [A,B^n]B + [A,B]\,B^n$$

$$= [A,B]\, n\, B^{n-1}\, B + [A,B]\,B^n = [A,B]\,(n+1)\, B^n$$

$\boxed{}$

A.2 $D(\alpha)$ can be expanded as

$$D(\alpha) = \exp\left[-\tfrac{1}{2}|\alpha|^2\right] \exp\left[\alpha a^\dagger\right] \exp\left[\alpha^* a\right]$$

Proof

Let's define

$$F(t) = \underbrace{\exp[tA]}_{e_A}\ \underbrace{\exp[tB]}_{e_B}\ \underbrace{\exp[-t(A+B)]}_{e_{-AB}}$$

with $[A,B]_2 = 0$ $\qquad [A,B]_n = [A,\underbrace{[\dots[A,B]]}_{n \text{ times}}]$

we have that

$$\frac{dF}{dt} = e_A A\, e_B\, e_{-AB} + e_A e_B \cancel{B}\, e_{-AB} - e_A e_B (A+\cancel{B})\, e_{-AB}$$

Using A.1

$$\frac{dF}{dt} = e_A [A, e_B]\, e_{-AB} = t\, e_A [A,B]\, e_B\, e_{-AB}$$

Remembering that $[A,[A,B]]=0$

$$\frac{dF}{dt} = t\,[A,B]\,F(t)$$

which, upon integration, gives

$$F(t)\big|_{t=1} = \exp\left\{\frac{1}{2}[A,B]\right\}$$

Comparing the preceding expression with the definition of $F(t)$, we have

$$\exp\left\{\frac{1}{2}[A,B]\right\} = \exp[A]\exp[B]\exp[-(A+B)]$$

posing $A = \alpha^* a$, $B = -\alpha a^\dagger$

$$\exp\left[-\frac{|\alpha|^2}{2}\right] = \exp\left[\alpha^* a\right] \exp\left[-\alpha a^\dagger\right] \exp\left[\alpha a^\dagger - \alpha^* a\right]$$

hence

$$D(\alpha) = \exp\left[-\frac{|\alpha|^2}{2}\right] \exp\left[\alpha a^\dagger\right] \exp\left[-\alpha^* a\right]$$
$$= \exp\left[\frac{|\alpha|^2}{2}\right] \exp\left[-\alpha^* a\right] \exp\left[\alpha a^\dagger\right]$$

A3

$$\langle \beta | \alpha \rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) + \alpha \beta^* \right]$$

<u>Proof</u>

since

$$|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \exp\left[\alpha a^\dagger\right] \exp\left[-\alpha^* a\right] |0\rangle \quad (\text{see A2})$$

and therefore

$$\langle \beta | \alpha \rangle = \langle 0 | \exp\left[\cancel{-\beta a^\dagger}\right] \exp\left[\beta^* a\right] \exp\left[\alpha a^\dagger\right] \exp\left[\cancel{-\alpha^* a}\right] | 0 \rangle \cdot \exp\left[-\frac{1}{2}\left(|\beta|^2 + |\alpha|^2\right)\right]$$

Taylor expanding $\exp\left[\alpha a^\dagger\right]$ and $\exp\left[\beta^* a\right]$ & using the orthogonality of number states, we get

$$\left(\langle 0| + \beta^* \langle 1| + \frac{\beta^{*2}}{2}\sqrt{2}\,\langle 2| + \frac{\beta^{*3}}{3!}\sqrt{3!}\,\langle 3| \dots\right)$$

$$\left(\dots + \frac{\alpha^3}{3!}\sqrt{3!}\,|3\rangle + \frac{\alpha^2}{2}\sqrt{2}\,|2\rangle + \alpha\,|1\rangle + |0\rangle\right)$$

and thus

$$\langle\beta|\alpha\rangle = \exp\left[-\frac{1}{2}\left(|\beta|^2 + |\alpha|^2\right)\right]\exp\left[\alpha\beta^*\right]$$

$\blacksquare$

A4

$$\mathbb{1} = \frac{1}{\pi}\int d^2\alpha \; |\alpha\rangle\langle\alpha|$$

Proof

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| =$$

$$\frac{1}{\pi}\sum_{nm}\frac{|n\rangle\langle m|}{\sqrt{n!\,m!}}\int d^2\alpha\, \exp\left[-|\alpha|^2\right]\alpha^{*m}\alpha^{n}$$

$$= \frac{1}{\pi}\sum_{nm}\frac{|n\rangle\langle m|}{\sqrt{n!\,m!}}\int_0^\infty dr\, r\, r^{n+m}\exp\left[-r^2\right]\underbrace{\int d\theta\, e^{i(n-m)\theta}}_{2\pi\,\delta_{nm}}$$

1)

$$= \frac{1}{\pi}\sum_{n}\frac{|n\rangle\langle n|}{n!}\;\frac{1}{2}\int_0^\infty d\varepsilon\, \exp\left[-\varepsilon\right]\varepsilon^{n}$$

$$= \frac{1}{\pi}\sum_{n}\frac{|n\rangle\langle n|}{\cancel{n!}}\;\frac{1}{2}\cancel{n!}\;2\pi = \sum |n\rangle\langle n| = \mathbb{1}$$

1) $$\frac{d^n}{d(-t)^n}e^{-t\varepsilon} = \varepsilon^n e^{-t\varepsilon}$$

$$\frac{d^n}{d(-t)^n}\int_0^\infty d\varepsilon\, e^{-t\varepsilon} = \frac{d^n}{d(-t)^n}\left[-\frac{e^{-t\varepsilon}}{t}\right]_0^\infty = \frac{d^n}{d(-t)^n}(t)^{-1} \overset{t\to 1}{=} n!$$

$\blacksquare$

4.5

$$a|n\rangle = \sqrt{n}\,|n\rangle$$

$$[a^\dagger a, a]|n\rangle = -a|n\rangle$$

$$a^\dagger a(a|n\rangle) - n(a|n\rangle) = -a|n\rangle$$

$$a^\dagger a(a|n\rangle) = (n-1)\,a|n\rangle$$

$$\|a|n\rangle\| = \langle n|a^\dagger a|n\rangle = \sqrt{n}$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

analog for $a^\dagger$

Coherence properties of the E.M. field.

How do we detect an E.M. field? ($\neq$ from proj. measurement)
We consider here a simple model based on Glauber's theory of detection [GLZ]: IDEAL PHOTON COUNTER $\Rightarrow$ it "clicks" when a photon is absorbed

$$E(\bar{r},t) = \sum_k i\left(\frac{\hbar\omega_k}{2\varepsilon_0}\right)^{\frac{1}{2}} \left\{ \underbrace{a_k\, u_k(\bar{r})\, \exp[-i\omega_k t]}_{E^+(\bar{r},t)\ \text{pos. freq}} - \underbrace{a_k^\dagger\, u_k^*(\bar{r})\, \exp[i\omega_k t]}_{E^-(\bar{r},t)\ \text{neg. freq}} \right\}$$

According to Fermi's golden rule, the transition probability for absorbing a photon at position $\bar{r}$ and time t is proportional to

$$T_{fi} = |\langle f|\, E^+(\bar{r},t)\,|i\rangle|^2$$

Since the detector measures only the total counting rate, we can sum over a complete set of final states $\left(\sum |f\rangle\langle f| = \mathbb{1}\right)$

A $$I(\bar{r},t) = \sum_f T_{fi} = \sum_f |\langle f|\, E^+(\bar{r},t)|i\rangle|^2$$
$$= \langle i|\, E^-(\bar{r},t)\; E^+(\bar{r},t)\,|i\rangle$$

$$E^-(\bar{r},t) = \left[E^+(\bar{r},t)\right]^\dagger$$

(N.B.: fields are normal-ordered $\Rightarrow \langle 0|\, E^-(\bar{r},t)\, E^+(\bar{r},t)|0\rangle = 0$)

We can generalise A, considering correlations at different spacetime points $x = (\bar{z}, t)$ and $x' = (\bar{z}', t')$, including the possibility of mixed states.

First-order correlation function

$$G^{(1)}(x, x') = \mathrm{Tr}\left\{ \rho\, E^-(x)\, E^+(x') \right\}$$

A further generalisation, allows us to define the n-th order correlation function

$$G^{(n)}(x_1 \dots x_n ; x_{n+1} \dots x_{2n})$$

$$= \mathrm{Tr}\left\{ \rho\, E^-(x_1) \dots E^-(x_n)\, E^+(x_{n+1}) \dots E^+(x_{2n}) \right\}$$

For instance, going back to the detector model, we could be interested in photon delayed coincidences, answering the question: what is the probability of detecting a photon at x and another photon at x'?

With a calculation similar to the one done for the single-photon detector, we get

$$\sum_f |\langle f | E^+(x') E^+(x) | i \rangle|^2 =$$

$$\langle i | E^-(x) E^-(x') E^+(x') E^+(x) | i \rangle$$

Properties of the correlation function

From the identity

B1 $$\mathrm{Tr}\{\rho A^{\dagger}A\} \geq 0$$

We can deduce a number of inequalities

a) $A = E^{+}(x) \quad \rightarrow \quad G^{(1)}(x,x) \geq 0$

b) $A = E^{+}(x_n) \dots E^{+}(x_1) \quad \rightarrow \quad G^{(n)}(x_1 \dots x_n; x_n \dots x_1) \geq 0$

c) $A = \sum_{j=1}^{n} \lambda_j E^{+}(x_j) \quad \rightarrow \quad \sum_{ij} \lambda_i \lambda_j \, G^{(1)}(x_i, x_j) \geq 0$

d) $A = \lambda_1 E^{+}(x_n) \dots E^{+}(x_1)$
$+ \lambda_2 E^{+}(x_{2n}) \dots E^{+}(x_{n+1}) \rightarrow$ verify ☆

$$G^{(n)}(x_1 \dots x_n, x_n \dots x_1)\, G^{(n)}(x_{n+1} \dots x_{2n}, x_{2n} \dots x_{n+1}) \geq$$

$$|G^{(n)}(x_1 \dots x_n, x_{n+1} \dots x_{2n})|^2$$

For 2 beams and with $x = (\bar{r}, 0)$, $x' = (\bar{r}, t)$

$$G^{(2)}_{11}(0)\, G^{(2)}_{22}(0) \geq [G_{12}(t)]^2$$

$G^{(n)}(x_i, x_j)$ in d) can be viewed as the matrix coefficients of a positive semi-definite quadratic form, implying

$$\det\left[G^{(n)}(x_i, x_j)\right] \geq 0$$

$n=1 \rightarrow$ c)

$n=2 \rightarrow \quad G^{(1)}(x_1,x_1)\; G^{1}(x_2,x_2) \geq |G^{(1)}(x_1,x_2)|^2$

$$\lambda_1^2 G_{11} + \lambda_1\lambda_2(G_{12}+G_{21}) + \lambda_2^2 G_{12} = (\lambda_1 \;\; \lambda_2)\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{pmatrix}\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} \geq 0$$

$\Rightarrow \det G > 0$

$\lambda_i = U_j x_i^j$

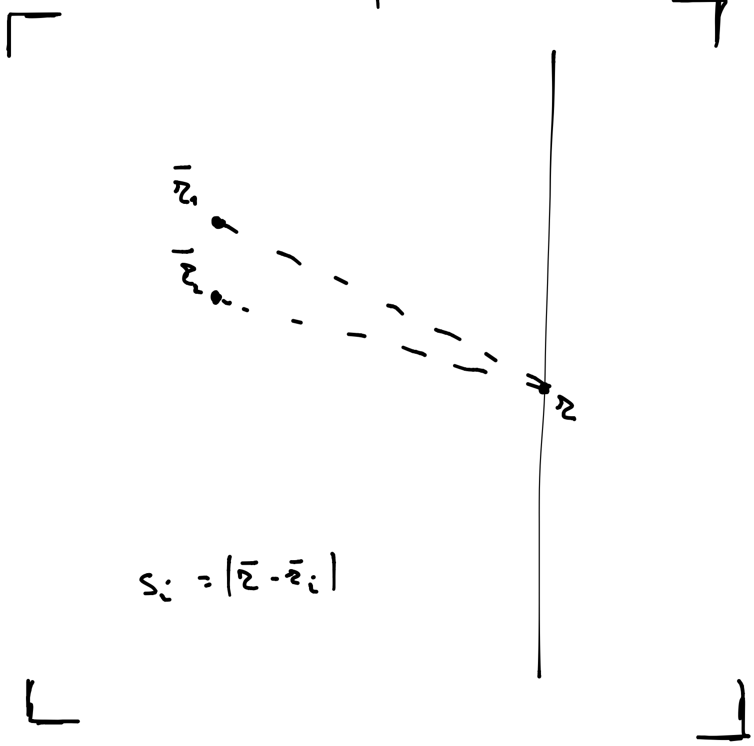
Sketch of proof. $\Rightarrow$

$$\lambda^i A_i^j \lambda_j = U^k x_k^i A_i^j U_e x_j^e = U^k x_k^i \varepsilon_i^i U_e x_i^e = U^k U_k \varepsilon_i^i$$

Classical optical interference experiment

The classical optical interference experiment corresponds to a measurement of the first-order correlation function. Let's discuss Young's double-slit experiment from this perspective.

The field produced at point $\bar{r}$ is the superposition of the fields originating at the two pinholes.



$$s_i = |\bar{r} - \bar{r}_i|$$

$$E^+(\bar{r},t) = E_1^+(\bar{r},t) + E_2^+(\bar{r},t)$$

where

$$E_i^+(\bar{r},t) = \frac{1}{s_i}\, E_i^+\left(\bar{r}_i,\, t - \frac{s_i}{c}\right) \exp\left[i\left(k - \frac{\omega}{c}\right)s_i\right]$$

Assuming $s_1 \simeq s_2 \simeq R$, $x_i = \left(\bar{r}_i,\, t - \frac{s_i}{c}\right)$, $\theta_i = \left(k - \frac{\omega}{c}\right)s_i$

$$E^+(\bar{r},t) = \frac{1}{R}\left[\, E^+(x_1)\,e^{i\theta_1} + E^+(x_2)\,e^{i\theta_2}\right]$$

the intensity observed on the screen $I = \mathrm{Tr}\{\rho\, E^-(\bar{r},t)\, E^+(\bar{r},t)\}$ becomes

$$I = G^{1}(x_1, x_1) + G^{1}(x_2, x_2) + 2\,\mathrm{Re}\left\{ G^{1}(x_1, x_2) \right\}$$

or

$$I = \underbrace{G^{1}(x_1, x_1)}_{\text{source 1}} + \underbrace{G^{1}(x_2, x_2)}_{\text{source 2}} + 2\,\underbrace{|G^{1}(x_1, x_2)|}_{\text{envelope of fringes}} \cos\underbrace{[\varphi(x_1, x_2)]}_{\text{fringes}}$$

If we define the visibility of the fringes $v = \dfrac{I_M - I_m}{I_M + I_m}$, we can express it as

$$v = \frac{2|G^{1}(x_1, x_2)|}{\left[G^{1}(x_1, x_1) + G^{1}(x_2, x_2)\right]}$$

which can be written in terms of normalised correlation function

$$g^{1}(x_1, x_2) = \frac{G^{1}(x_1, x_2)}{\left[G^{1}(x_1, x_1)\, G^{1}(x_2, x_2)\right]^{\frac{1}{2}}}$$

and $I_1 = G^{1}(x_1, x_1)$, $I_2 = G^{1}(x_2, x_2)$ as

$$v = |g^{1}| \frac{2(I_1 I_2)^{\frac{1}{2}}}{I_1 + I_2}$$

For $I_1 = I_2$, we have

$$v = |g^1|$$

Let's use what we know about the correlation functions i.e.:

$$G^1(x_1, x_1)\, G^1(x_2, x_2) \geq |G^1(x_1, x_2)|^2$$

meaning that $|g^1| \leq 1$, and thus the condition for (first-order) optical coherence translate to maximum fringe visibility.
On more general grounds (perfect) n^{th} order coherence can be defined as the condition for which the n^{th} order correlation function factorizes.

$$\text{A} \qquad G^n(x_1 \ldots x_n, x_{n+1} \ldots x_{2n}) = \mathcal{E}^-(x_1) \ldots \mathcal{E}^-(x_n) \cdot \mathcal{E}^+(x_{n+1}) \ldots \mathcal{E}^+(x_{2n})$$

for $n=1$ this gives back the condition $|g^1(x_1, x_2)| = 1$.
It is trivial to check that for coherent states the property A holds - here lies the original motivation (and definition) of f.c.s.

Let's now analyse the double-slit experiment in a bit more detail for 3 different scenarios:

a. Double-slit / independent light sources for coherent light
b. Double-slit experiment for 1 photon.
c. Double-slit / independent light sources for thermal light

Foreword

Let's rewrite $E^+(\vec{r},t)$ as

$$E^+(\vec{r},t) = f(\vec{r},t)\left(a_1 e^{ikr_1} + a_2 e^{ikr_2}\right)$$

$$f(r,t) = i\left(\frac{\hbar\omega}{2}\right)^{1/2} \frac{\hat{e}_k}{(4\pi L)^{1/2}} \frac{1}{R} e^{-i\omega t}$$

The intensity becomes

$$I(\bar{z},t) = |f(\bar{z},t)|^2 \Big[\mathrm{Tr}\{\rho\, a_1^\dagger a_1\} + \mathrm{Tr}\{\rho\, a_2^\dagger a_2\} + 2\, |\mathrm{Tr}\{\rho\, a_1^\dagger a_2\}| \cos\phi \Big]$$

$$\phi = k(s_1 - s_2) + \varphi \qquad \varphi = \mathrm{Arg}\big[\mathrm{Tr}\{\rho\, a_1^\dagger a_2\}\big]$$

a. Let's consider a coherent field, e.g. generated by a laser above threshold incident on two slits

$$|\text{coh. field}\rangle = |\alpha_1\rangle|\alpha_2\rangle$$

Since $|\alpha_1\rangle|\alpha_2\rangle$ is a product state, it can either represent two independent light beams, or be generated, for instance, by the displacement operator $D(\alpha) = \exp(\alpha b^\dagger - \alpha^* b)$, with $b^\dagger = \eta_1 a_1^\dagger + \eta_2 a_2^\dagger$, leading to

$$|\alpha_1 \alpha_2\rangle = D(\alpha)\,|0\rangle = |\eta_1\alpha\,,\, \eta_2\alpha\rangle$$

(single coherent source)

In both cases the intensity pattern produced by the state $|\alpha_1 \alpha_2\rangle$ is given by

$$I = |\alpha_1|^2 + |\alpha_2|^2 + 2|\alpha_1\alpha_2| \cos\left[K(s_1 - s_2) + Arg(\alpha_1^* \alpha_2)\right]$$

This result is obtained calculating the values of $Tr\{\rho\, a_1^\dagger a_1\}$, $Tr\{\rho\, a_2^\dagger a_2\}$, $Tr\{\rho\, a_1^\dagger a_2\}$ explicitly for $\rho = |\alpha_1\rangle\langle\alpha_1| \otimes |\alpha_2\rangle\langle\alpha_2|$.
The same strategy will be used for the single-photon and the thermal source cases.

In the experiment, in order to observe the interference between two independent beams, the relative phase $Arg(\alpha_1^* \alpha_2)$ should be slowly varying (time average of the cos term), meaning that the two sources should be rendered insensitive enough to potential external noise sources.

b. Actual double-slit experiment for 1 photon.

There are interference fringes for 1 photon.
$\Rightarrow$ Original argument by Dirac: the fringes result from the interference between the probability amplitude of the photon to go through either slit. This is understandable from a path-integral point of view (∞-slits idea).
A photon $b^\dagger|0\rangle$ can be represented as a superposition of a photon going through either slit 1 and 2

$$|1ph\rangle = b^\dagger|0\rangle = \frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger)|0\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle)$$

Leading to

$$\left[Tr\{\rho\, a_i^\dagger a_i\} = \frac{1}{2} \qquad Tr\{\rho\, a_1^\dagger a_2\} = \frac{1}{2} \right]$$

$$I = 1 + \cos\left[k(s_1 - s_2)\right]$$

which coincides with the expression obtained for the coherent source if $\alpha_1 = \alpha_2 = 1$.

c. Double slit / indep. sources for thermal light.

$$\left[P_n = \frac{\exp[-\beta\hbar\omega n]}{Z} \right.$$

Thermal state for photons

$$Z = \sum_n \exp[-\beta\hbar\omega n]$$

$$\rho = \sum_n p_n \, |n\rangle\langle n|$$

$$\left. = \left(1 - \exp[-\beta\hbar\omega]\right)^{-1} \right]$$

$$\rho = \sum_n P_n \left(\frac{1}{\sqrt{n+1}} \sum_{n_a} |n_a\rangle_1 |n-n_a\rangle_2\right)\left(\frac{1}{\sqrt{n+1}} \sum_{n_a} {}_2\langle n-n_a|\, {}_1\langle n_a|\right)$$

$$Tr\{\rho\, a_i^\dagger a_i\} = \sum_n \frac{1}{n+1} P_n \sum_{n_a n_b} {}_2\langle n-n_b|\, {}_1\langle n_b|\, a_i^\dagger a_i\, |n_a\rangle_1 |n-n_a\rangle_2$$

$$= \sum_n \frac{1}{n+1} P_n \sum_{n_a} n_a = \sum_n P_n \frac{\cancel{(n+1)}\, n}{\cancel{(n+1)}\, 2} = \frac{\langle n\rangle}{2}$$

$n_b = n_a + 1$

$$\mathrm{Tr}_2\left\{\rho a_1^\dagger a_2\right\} = \sum_n \frac{P_n}{n+1} \sum_{n_a} {}_2\langle n - n_a - 1|\,{}_1\langle n_a+1|\, a_1^\dagger a_2 \,|n_a\rangle_1\, |n - n_a\rangle_2$$

$$= \sum_n \frac{P_n}{n+1} \sum_{n_a} \sqrt{n - n_a}\,\sqrt{n_a + 1} \simeq \sum_n P_n \sqrt{1 - \frac{n_a}{n}}\,\sqrt{\frac{n_a+1}{n}}$$

$$\sum_n P_n \; n \int_0^1 \sqrt{(1-x)x}\; dx = \frac{\pi}{8}\langle n\rangle$$

$x = \sin^2\theta \qquad dx = 2\sin\theta\cos\theta\, d\theta$

$$\int_0^{\pi/2} \frac{\sin^2 2\theta}{2}\, d\theta = \int_0^{2\pi} \frac{1 - \cos 4}{16}\, d\varphi = \frac{1}{16}\varphi\,\Big|_0^{2\pi} + \sin\varphi\,\Big|_0^{2\pi}$$

$$= \frac{\pi}{8}$$

Appendix B

Proof of B.1

ρ can be written as

$$\rho = \sum \lambda_i |\varphi_i\rangle\langle\varphi_i| \qquad \text{with} \qquad \lambda_i > 0$$

Thus

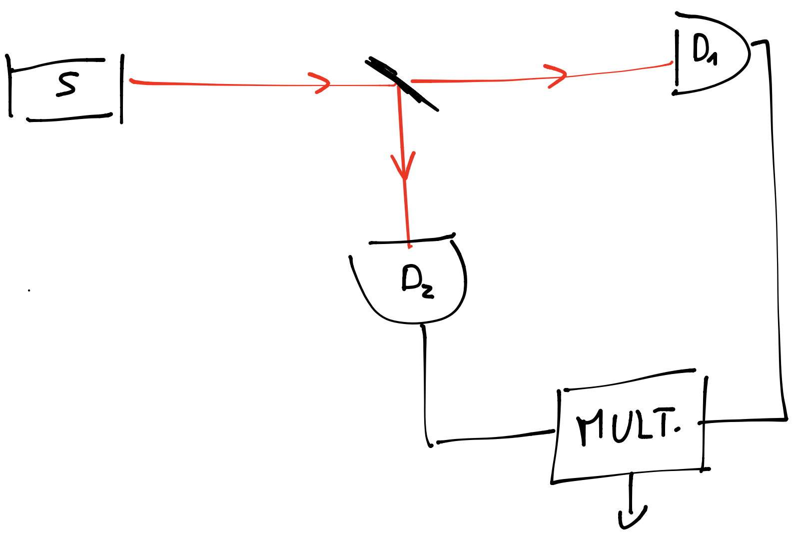
$$\mathrm{Tr}\{\rho A^\dagger A\} = \mathrm{Tr}\Big\{\sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i| A^\dagger A\Big\} = \sum_{ij} \lambda_i \overbrace{\langle\varphi_j|\varphi_i\rangle}^{\delta_{ij}} \langle\varphi_i|A^\dagger A|\varphi_j\rangle$$

$$= \sum_i \lambda_i \langle\varphi_i|A^\dagger A|\varphi_i\rangle = \sum_i \lambda_i \, \| A|\varphi_i\rangle \|^2 \geq 0$$

3. Photon correlation measurements

Hanbury-Brown & Twiss experiment [HB] was the first one outside the 1-photon domain.

The idea is: what is the probability of detecting a photon in D_1 @ t_1 (x_1) and another in D_2 @ t_2 (x_2)?



$$G^{(12)}(x_1 x_2, x_2 x_1) = \langle E^-(x_1) E^-(x_2) E^+(x_1) E^+(x_2) \rangle$$

$$= \langle : I(t)\, I(t+\tau) : \rangle \propto \langle : n(t)\, n(t+\tau) : \rangle$$

normal ordering ⊢

From now on $\vec{r}_1 = \vec{r}_2$ $\quad t_1 = t = t_2 - \tau$ → what is the probability of detecting two photons separated by time t?

It is useful to introduce the normalized 2nd order correlation function

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{|G^{(1)}(0)|^2}$$

For a coherent field, as anticipated

$$G^{(2)}(\tau) = \mathcal{E}^-(t)\, \mathcal{E}^-(t+\tau)\, \mathcal{E}^+(t+\tau)\, \mathcal{E}^+(t) \overset{\tau\to 0}{=} \left[G^{(1)}(0) \right]^2$$

hence

$$g^{(2)}(0) = 1$$

For a fluctuating classical field we may introduce a probability distribution $P(\varepsilon)$ of the field $E^+(\varepsilon, t)$ having amplitude ε, where

$$E^+(\varepsilon, t) = -i \left(\frac{\hbar \omega_0}{2\varepsilon_0 V} \right)^{1/2} \varepsilon\, e^{-i\omega t}$$

$G^{(2)}(\tau)$ becomes

$$G^{(2)}(\tau) = \int_0^{\varepsilon} P(\varepsilon)\; E^-(\varepsilon, t)\, E^-(\varepsilon, t+\tau)\, E^+(\varepsilon, t+\tau)\, E^+(\varepsilon, t)\, d\varepsilon$$

$$\int P_\varepsilon\, \varepsilon^4 - 2 \langle \varepsilon^2 \rangle^2 + \langle \varepsilon^2 \rangle^2 + \langle \varepsilon^2 \rangle^2$$

for $\tau = 0$

3.1 $$g^{(2)}(0) = 1 + \frac{\int_0^{\varepsilon} d\varepsilon\; P(\varepsilon) \left(|\varepsilon|^2 - \langle |\varepsilon|^2 \rangle \right)^2}{\langle |\varepsilon|^2 \rangle^2}$$

$\Rightarrow \quad g^{(2)}(0) \geq 1$ for a classical field (coh. field $g^{(2)}(\tau) = 1$)

For a classical field obeying Gaussian statistics, (with zero mean amplitude) and no phase-dependent fluctuations, we have

↳ Wick's theorem

$$\langle E^-(\varepsilon,t)\, E^-(\varepsilon,t+\tau)\, E^+(\varepsilon,t+\tau)\, E^+(\varepsilon,t)\rangle =$$
$$\langle E^-(\varepsilon,t)\, E^+(\varepsilon,t)\rangle\langle E^-(\varepsilon,t+\tau)\, E^+(\varepsilon,t+\tau)\rangle +$$
$$\langle E^-(\varepsilon,t)\, E^+(\varepsilon,t+\tau)\rangle\langle E^-(\varepsilon,t+\tau)\, E^+(\varepsilon,t)\rangle$$

thus

$$G^{(2)}(t) = G^{(1)}(0)^2 + |G^{(1)}(\tau)|^2$$

and

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2.$$

Since $G^{(1)}(\tau)$ is the inverse Fourier transform of the spectrum of the field

$$G^{(1)}(\tau) = \int \frac{d\omega}{2\pi} S(\omega) \exp[-i\omega\tau]$$

we have

$$g^{(2)}(\tau) = 1 + e^{-\gamma t}$$

for a field with Lorentzian spectrum, and

$$g^{(2)}(\tau) = 1 + e^{-\gamma^2 t^2}$$

measurement of $g^{(2)}(\tau)$ is a way to reconstruct the spectrum (photon correlation statistics) ← REF 3.10 Walls & Milburn

For $\tau \gg \tau_c$ (τ_c correlation time) the photons tend to become uncorrelated, hence $g^{(2)}(\infty) = 1$.

The fact that $g^{(2)}(0) > 1$ for classical light reflects the fact that a photon triggers the photodetector during a high intensity fluctuation and thus another photon is detected arbitrarily soon: this effect is termed photon bunching.

Let's now calculate $g^{(2)}(0)$ for some quantum mechanical fields.

In general, we have

$$g^{(2)}(0) = \frac{\mathrm{Tr}\{\rho a^\dagger a^\dagger a a\}}{\mathrm{Tr}\{\rho a^\dagger a\}^2} = \frac{\mathrm{Tr}\{\rho[(a^\dagger a)^2 - a^\dagger a]\}}{\mathrm{Tr}\{\rho a^\dagger a\}^2}$$

a. Thermal state

A single-mode thermal state is defined as a mixed state

$$\rho = \sum p_n |n\rangle\langle n|$$

where $|n\rangle$ are number states and

$$p_n = Z^{-1} \exp[-\beta\hbar\omega n] \qquad Z = \sum_n \exp[-\beta\hbar\omega n] = (1 - \exp[-\beta\hbar\omega])^{-1}$$

therefore

$$\bar{n} = \frac{\partial_\alpha Z}{Z} \qquad \bar{n} = \frac{e}{1-e} \qquad p_n = e^n(1-e) = e^{n+1}\frac{1-e}{e} = \frac{\bar{n}^n}{1+\bar{n}^{n+1}}$$

$$\bar{n} = e(1+\bar{n}) \qquad e = \frac{\bar{n}}{1+\bar{n}}$$

$$g^{(2)}(0) = (1 - \exp[-\beta\hbar\omega])^{-1} \frac{\mathrm{Tr}\{|n\rangle\langle n| \exp[-\beta\hbar\omega n][(a^\dagger a)^2 - a^\dagger a]\}}{\mathrm{Tr}\{|n\rangle\langle n| \exp[-\beta\hbar\omega n]\, a^\dagger a\}^2}$$

$$p_n^{coh} = \frac{\bar{n}^n \exp[-\bar{n}]}{n!}$$

$$= (1 - \exp[\beta\hbar\omega])^{-1} \frac{\sum_n \exp[\beta\hbar\omega n](n^2 - n)}{\left(\sum_n n \exp[-\beta\hbar\omega n]\right)^2} +$$

The previous expression can be written in terms of Z and its derivatives with respect to $\alpha = -\beta\hbar\omega$ as

$$g^{(2)}(0) = -Z \frac{\partial_\alpha^2 Z + \partial_\alpha Z}{(\partial_\alpha Z)^2} = 2$$

Thermal source will thus exhibit bunching and thus, according to 3.1 exhibit photon correlation properties compatible with a classical field.

b. Coherent state

$$g^{(2)}(0) = \frac{\langle a^+ a^+ a a \rangle}{\langle a^+ a \rangle^2} = \frac{|\alpha|^4}{|\alpha|^4} = 1$$

c. Fock state

$$g^{(2)}(0) = \frac{\langle a^+ a^+ a a \rangle}{\langle a^+ a \rangle^2} = \frac{\langle a^+ a\, a^+ a \rangle - \langle a^+ a \rangle}{\langle a^+ a \rangle^2} = 1 - \frac{1}{n}$$

Fock states exhibit photon correlation properties which cannot be described classically (3.1), for Fock states we have that $g^{(2)}(0) < g^{(2)}(\tau \gg \tau_c)$ → photon antibunching ("granularity" of the radiation).

The discussion about photon (bunching/antibunching) can be related to the statistics of the state. For a stationary state, it is possible to relate bunching/antibunching character of $g^{(2)}(\tau)$ to the super/sub-Poissonian character of the state

$$V(N) - \langle N \rangle = \frac{\langle N \rangle^2}{T^2} \int_{-T}^{T} d\tau \, (T - |\tau|) \left[g^{(2)}(\tau) - 1 \right] \qquad (3.2)$$

(where T is the observation time interval)

N.B.: $g^{(2)}(\tau) < 1 \ \forall \tau \Rightarrow$ sub Poissonian statistics, but a field for which $g^{(2)}(0) < g^{(2)}(\tau)$, can have super-Poissonian statistics over some timescale.

Squeezed state & statistics

We evaluate $V(n) - \langle n \rangle$ for a squeezed state.

With

$$\langle \alpha, \zeta | a^\dagger a^\dagger a a | \zeta, \alpha \rangle = \langle 0 | (\alpha^* + c^* a^\dagger + s^* a)^2 (\alpha + c a + s a^\dagger)^2 | 0 \rangle$$

$$= \langle 0 | \left[\alpha^{*2} + \cancel{c^{*2} a^{\dagger 2}} + s^{*2} a^2 + 2\alpha^* \cancel{c^* a^\dagger} + 2\alpha^* s^* a + c^* s^* (\cancel{2n}+1)\right] \left[\alpha^2 + \cancel{c^2 a^2} + s^2 a^{\dagger 2} + 2\alpha c \cancel{a} + 2\alpha s a^\dagger + c s (\cancel{2n}+1)\right] | 0 \rangle$$

$$= \langle 0 | (\alpha^{*2} + s^{*2} a^2 + 2\alpha^* s^* a + c^* s^*)(\alpha^2 + s^2 a^{\dagger 2} + 2\alpha s a^\dagger + c s) | 0 \rangle$$

$$= |\alpha|^4 + |s|^4 + 4|\alpha|^2 |s|^2 + |c s|^2 + c(\alpha^{*2} s + \alpha^2 s^*) - 2|\alpha|^2 |s|^2$$

Remembering that

$$\bar{n} = \langle \alpha, \zeta | n | \zeta, \alpha \rangle = |\alpha|^2 + |s|^2$$

we get

$$V(n) - \langle n \rangle = |\alpha|^4 + |s|^4 + 4|\alpha|^2 |s|^2 + |c s|^2 + c(\alpha^{*2} s + \alpha^2 s^*) - (|\alpha|^2 + |s|^2)^2$$

$$= 2|\alpha|^2 |s|^2 + |c s|^2 (\alpha^{*2} + \alpha^2 + 1)$$

Assuming that $\zeta = z \in \mathbb{R}$, $\alpha = |\alpha| e^{i\theta}$, we have

$$\frac{V(n) - \bar{n}}{\bar{n}^2} = \frac{|\alpha|^2 \left(\operatorname{ch} 2z - \operatorname{sh} 2z \cos 2\theta - 1 \right) + 2 \operatorname{sh}^2 z \, \operatorname{ch}^2 z}{\left(|\alpha|^2 + \operatorname{sh}^2 z \right)^2}$$

When $\theta = \frac{\pi}{2}$ (squeezing out of phase with the complex amplitude)

$$V(n) - \bar{n} = |\alpha|^2 \left(e^{2z} - 1 \right) + 2 \operatorname{sh}^2 z \, \operatorname{ch}^2 z$$

meaning that a state with increased amplitude fluctuations has, as expected, super-Poissonian statistics.

If $\theta = 0$

$$V(n) - \bar{n} = |\alpha|^2 \left(e^{-2z} - 1 \right) + 2 \operatorname{sh}^2 z \, \operatorname{ch}^2 z$$

the squeezing is in phase with the complex amplitude, and the first term corresponds to a reduction of the fluctuations with respect to a Poissonian distribution, the second term corresponds to the fluctuations of the squeezed vacuum. If $|\alpha|^2 \gg 2 \operatorname{sh}^2 z \, \operatorname{ch}^2 z$ this is an amplitude squeezed state with subPoissonian photon statistics.

Appendix C

Proof of 3.2

$V(M)$ is defined as

$$\langle M^2 \rangle - \langle M \rangle^2$$

$$\langle M^2 \rangle = \frac{1}{T^2} \int_0^T dt_1 \int_0^T dt_2 \, \langle E^-_{t_2} E^+_{t_2} E^-_{t_1} E^+_{t_1} \rangle$$

$$\frac{1}{T^2} \int_0^T dt_1 \int_0^T dt_2 \, G^{(2)}(t_2 - t_1) + T \, \delta\left(\frac{t_2 - t_1}{T}\right) G^{(1)}(t_1)$$

$$= \frac{1}{T^2} \int_0^T dt_1 \int_0^T dt_2 \, G^{(2)}(t_2 - t_1) + \frac{1}{T} \int_0^T dt_1 \, G^{(1)}(t_1)$$

$$= \frac{1}{T^2} \int_0^T dt_1 \int_0^T dt_2 \, G^{(2)}(t_2 - t_1) + G^{(1)}(0)$$

$$\langle M \rangle = G^{(1)}(0)$$

$$\langle M \rangle^2 = G^{(1)}(0)^2 = \frac{1}{T^2} \int_0^T dt_1 \int_0^T dt_2 \, G^{(1)}(0)^2$$

$$\langle M^2 \rangle - \langle M \rangle^2 = \frac{\langle M \rangle^2}{T^2} \int_0^T dt_1 \int_0^T dt_2 \left[g^{(2)}(t_2 - t_1) - 1 \right] + \langle M \rangle$$

$$V(M) - \langle M \rangle = \frac{\langle M \rangle^2}{T^2} \int_0^T dt_1 \int_0^T dt_2 \left[g^{(2)}(t_2 - t_1) - 1 \right]$$

$$t = t_1 + t_2 \qquad \tau = t_2 - t_1$$

$$dt\, d\tau = \underbrace{\begin{vmatrix} \frac{\partial t}{\partial t_1} & \frac{\partial t}{\partial t_2} \\ \frac{\partial \tau}{\partial t_1} & \frac{\partial \tau}{\partial t_2} \end{vmatrix}}_{\to 2} dt_1\, dt_2$$

$$= \frac{1}{2} \frac{\langle N \rangle^2}{T^2} \int_{-T}^{T} d\tau \int_{|\tau|}^{2T-|\tau|} dt \left[g^{(2)}(\tau) - 1 \right]$$

$$= \frac{\langle N \rangle^2}{T^2} \int_{-T}^{T} d\tau \left[g^{(2)}(\tau - 1) \right] (T - |\tau|)$$