Phase space formulation of quantum mechanics.
(nano intro)

In classical mechanics the expectation value of a given observable a is defined as

$$\langle A \rangle = \int d\bar{q}\, d\bar{p}\; P(\bar{q},\bar{p})\; A(\bar{q},\bar{p}) \qquad (4.1)$$

where $P(\bar{q},\bar{p})$ - with $\bar{q} = \bar{q}_1, \dots \bar{q}_M$, $\bar{p} = \bar{p}_1, \dots \bar{p}_M$ - represents the probability for the system to occupy the phase-space point $(\bar{q},\bar{p})$ and $A(\bar{x},\bar{p})$ the value of the observable a at that point.
In a quantum context, due to the Heisenberg uncertainty principle, the concept of phase-space point is ill-defined.
How can we introduce a formulation of Q.M. treating position and momentum on equal footing as a bona-fide quantum version of (4.1) would require?
The canonical quantisation procedure in either position $\left(\hat{q} \to q\,,\ \hat{p} \to -i\hbar \frac{\partial}{\partial q}\right)$ or momentum $\left(\hat{p} \to p,\ \hat{q} \to -i\hbar \frac{\partial}{\partial p}\right)$ representation manifestly violates this requirement.
In order to define the quantum-mechanical analogue of the classical phase space, we introduce a mapping between functions in the quantum phase space and operators in the Hilbert space.

This mapping is provided by the Weyl Wigner transform, where the Weyl mapping associates a function $a(q,p)$ defined in the quantum phase space, to an Hilbert space operator $\hat{A}[\hat{q},\hat{p}]$ through the relation (Weyl quantisation)

Kernel

$$\hat{A}[\hat{q},\hat{p}] = \frac{1}{(2\pi)^2}\int dq\, dp\, dk\, d\tau \; a(q,p)\,\phi(k,\tau)\,\exp\left\{i\left[k(\hat{q}-q)+\tau(\hat{p}-p)\right]\right\} \qquad (4.2)$$

Conceptually, eq. 4.2 can be interpreted as the combination of a Fourier transform with a ("quantum") inverse F.T. The inverse transform of 4.2 defining $a(q,p)$ from $\hat{A}[\hat{q},\hat{p}]$ is termed Wigner transform (see D1 for a proof)

$$a(q,p) = \frac{1}{(2\pi)}\int dk\, d\tau\, d\xi \; \frac{1}{\phi(k,\tau)}\exp[ikq]\exp[ip\tau]\exp[-ik\xi]\,\langle \xi - \tfrac{\tau}{2}|\hat{A}|\xi + \tfrac{\tau}{2}\rangle \qquad (4.3)$$

Schematically we have

characteristic function

$$\text{(4.4a)} \qquad f(q,p) \xRightarrow[\substack{q\to k\\ p\to \tau}]{FT} \chi(k,\tau) \xRightarrow[\substack{k\to\hat{q}\\ \tau\to\hat{p}}]{\text{"}q\text{" } FT^{-1}} \hat{F}[\hat{q},\hat{p}]$$

"q" ph. space

Op. on H. space

$$\text{(4.4b)} \qquad f(q,p) \xLeftarrow{FT^{-1}} \chi(k,\tau) \xLeftarrow{\text{"}q\text{" } FT} \hat{F}[\hat{q},\hat{p}]$$

Why do we need all this in the context of q. optics?
It turns out that a (class of) Wigner transforms of the density matrix provide a particularly useful characterisation of the state of an electromagnetic field, providing some useful insight on the quantum nature (or lack thereof) of the state.

Following 4.4.b we have that the "q"-FT of the density matrix $\hat{\rho}$ can be written as

$$\chi(\eta) = \mathrm{Tr}\left\{\hat{\rho}\exp\left[-ik\hat{q} - iz\hat{p}\right]\right\} = \mathrm{Tr}\left\{\hat{\rho}\exp\left[\eta a^{\dagger} - \eta^{*} a\right]\right\}$$

with $\eta = \frac{1}{\sqrt{2}}(z - ik)$. $\chi(\eta)$ is uniquely determined by the density matrix $\hat{\rho}$ and it is called characteristic function.
Furthermore it is possible to define normally (χ_N) and antinormally (χ_A) ordered characteristic functions

$$\chi_N(\eta) = \mathrm{Tr}\left\{\rho\, e^{\eta a^{\dagger}} e^{-\eta^{*} a}\right\}$$

$$\chi_A(\eta) = \mathrm{Tr}\left\{\rho\, e^{-\eta^{*} a} e^{\eta a^{\dagger}}\right\} \qquad 4.5$$

which, since

$$D(\eta) = \exp\left[-\frac{|\eta|^2}{2}\right] \exp[\eta a^\dagger] \exp[-\eta^* a]$$

$$= \exp\left[\frac{|\eta|^2}{2}\right] \exp[-\eta^* a] \exp[\eta a^\dagger]$$

can be written as

$$\chi_N(\eta) = \exp\left(\frac{|\eta|^2}{2}\right) \chi(\eta)$$

4.6

$$\chi_A(\eta) = \exp\left(-\frac{|\eta|^2}{2}\right) \chi(\eta)$$

The subsequent $[FT]^{-1}$ in 4.4.b of $\chi(\eta)$, $\chi_N(\eta)$ and $\chi_A(\eta)$ leads to the definition of the Wigner function $W(q,p)$ to the P function $P(q,p)$ and the Q function $Q(q,p)$, respectively.

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\eta\, \chi(\eta) \exp[\eta^*\alpha - \eta\alpha^*] \quad a.$$

4.7

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\eta\, \chi_N(\eta) \exp[\eta^*\alpha - \eta\alpha^*] \quad b.$$

$$Q(\alpha) = \frac{1}{\pi^2}\int d^2\eta\, \chi_A(\eta) \exp[\eta^*\alpha - \eta\alpha^*] \quad c.$$

From 4.7 a. b. c. it is possible to see how $W(\alpha)$, $P(\alpha)$ and $Q(\alpha)$ correspond to the Wigner-Weyl transform of the density operator with kernel $\phi_W = 1$, $\phi_P = \frac{1}{\pi}\exp\left[-\frac{1}{4}(\tau^2+\kappa^2)\right]$ and $\phi_Q = \frac{1}{\pi}\exp\left[\frac{1}{4}(\tau^2+\kappa^2)\right]$ respectively (see 0.2).

$P(\alpha)$

The funtion $P(\alpha)$ can be used to represent the density matrix in terms of coherent states, and, in fact was originally introduced in this form by Glauber [G12] and Sudarshan [Su] (see D3 for a proof)

$$4.7 \qquad \rho = \int d^2\alpha \; P(\alpha) \, |\alpha\rangle\langle\alpha|$$

The function $P(\alpha)$ can be interpreted as a quasi-probability distribution: i.e. it represents a probability distribution for classical fields. This property is related to the fact that for $|\alpha - \alpha'| \gg 1$, coherent states are approximately orthogonal, and thus $|\alpha\rangle\langle\alpha|$ represents a projection operator over (almost) orthogonal states. In general, however, $P(\alpha)$ can be negative or highly singular.

P representation for:

1. Coherent state:

$$\rho = |\alpha_0\rangle\langle\alpha_0| \Rightarrow P(\alpha) = \delta(\alpha - \alpha_0)$$

2. Thermal state

$$\rho = (1 - \exp[-\hbar\omega\beta]) \sum_n \exp[-\hbar\omega\beta n] \, |n\rangle\langle n|$$

$$= \sum_n \frac{\bar{n}^{\,n}}{(1+\bar{n})^{n+1}} \, |n\rangle\langle n|$$

$$P(\alpha) = \frac{1}{\pi \bar{n}} \exp\left[- \frac{|\alpha|^2}{\bar{n}} \right]$$

D.4 for a proof.

3. Number state

$$P(\alpha) = \sum_n \langle n|\hat{\rho}|k\rangle \frac{\sqrt{n!\,k!}}{2\pi|\alpha|\,(n+k)!}$$

$$\exp\left[|\alpha|^2 - i(n-k)\theta\right] \left[\left(- \frac{\partial}{\partial|\alpha|} \right)^{n+k} \delta(|\alpha|) \right] \;!$$

P representation is convenient for evaluating normally-ordered products such as

$$\langle a^{\dagger m} a^{m} \rangle = \int d^2\alpha \; P(\alpha)\, \alpha^{*m} \alpha^{m}$$

since $\rho = \int d^2\alpha \; P(\alpha)\, |\alpha\rangle\langle\alpha|$, we have

$$\mathrm{Tr}\left\{ \rho\, a^{\dagger m} a^{m} \right\} = \int d^2\alpha \, \langle \alpha | a^{\dagger m} a^{m} | \alpha \rangle$$

For instance, we can evaluate $g^{(2)}(0)$ as

$$g^{(2)}(0) = \frac{\mathrm{Tr}\{\rho a^\dagger a^\dagger a a\}}{\mathrm{Tr}\{\rho a^\dagger a\}} = \frac{\int d^2\alpha \; P(\alpha)\, |\alpha|^4}{\int d^2\alpha \, P(\alpha)\, |\alpha|^2}$$

$$= 1 + \frac{\int d^2\alpha \; P(\alpha) \left[|\alpha|^2 - \langle |\alpha|^2 \rangle \right]^2}{\left[\int d^2\alpha \; P(\alpha)\, |\alpha|^2 \right]^2}$$

This expression is formally analogous to the classical one given in the previous chapter, but now no constraint is placed on the value of $P(\alpha)$, thus allowing $g^{(2)}(0) < 1$, i.e. photon antibunching.

Wigner function

Properties of the Wigner function

1. It's real $\left(\langle q - \frac{\tau}{2} | \hat{\rho} | q + \frac{\tau}{2} \rangle = \langle q + \frac{\tau}{2} | \hat{\rho} | q - \frac{\tau}{2} \rangle^* \right)$

2. Its marginals with respect to q, p are are the probability distributions for p and q respectively [see 02]

$$\int dp \, W(q,p) = \int d\tau \, \langle q + \tfrac{\tau}{2} | \hat{\rho} | q - \tfrac{\tau}{2} \rangle \int \frac{dp}{2\pi} \exp[ip\tau]$$

$$= \int d\tau \, \delta(\tau) \, \langle q + \tfrac{\tau}{2} | \hat{\rho} | q - \tfrac{\tau}{2} \rangle = \langle q | \hat{\rho} | q \rangle$$

and, analogously,

$$\int dq \, W(q,p) = \langle p | \hat{\rho} | p \rangle$$

(from the "symmetry" between p and q)

3. The overlap between two states can be evaluated through the Wigner function as

$$\mathrm{Tr}\{\hat\rho_1 \hat\rho_2\} = |\langle \psi_1|\psi_2\rangle|^2 =$$
$$= 2\pi \int dq \int dp\, W(q,p)\, W_2(q,p)$$

4. In presence of (at most) an harmonic potential its dynamics is classical

$$\partial_t W(q,p) = -\frac{p}{m}\,\partial_q W(q,p)$$ /- Not proven here.

Examples of Wigner functions

1. Coherent state

$$W(\alpha) = \frac{2}{\pi} \exp\left[-2|\alpha-\alpha_0|^2\right]$$

Proof from $P(\alpha)$, see below

2. Squeezed state

$$W(x_1,x_2) = \frac{2}{\pi} \exp\left[-\left(x_1^2/2\; e^{2r} - x_2^2/2\; e^{-2r}\right)\right]$$

③ Thermal state

$$W(\alpha) = \frac{1}{\pi(\bar{n}+\frac{1}{2})} \exp\left[-\frac{|\alpha|^2}{\bar{n}+\frac{1}{2}}\right]$$

④ Number state

$$W(\alpha) = \frac{2}{\pi} (-1)^n L_n\left(4|\alpha|^2\right) \exp\left[-2|\alpha|^2\right]$$

States 1-3 belong to the class of Gaussian states in the context of continuous variable quantum in the context of continuous variables quantum information (see [Ad]). Moreover, we know that a (in general multivariate) Gaussian is completely characterized by its first two moments, hence the "uncertainty" ellipses that we were drawing when discussing coherent and squeezed states correspond, in the appropriate quadratures, to the variances associated with the (Gaussian) Wigner function of coherent and squeezed states (see also below)

The Wigner function is related to $P(\alpha)$ by the following relation

$$W(\alpha) = \frac{2}{\pi} \int d^2\alpha \; P(\beta) \exp\left[-2|\beta-\alpha|^2\right] \qquad (4.8)$$

i.e. the W. function is given by the convolution of the P function with a Gaussian, hence representing, in some sense, a "smeared" version of the possibly highly singular $P(\alpha)$.

Q - function

The most "tame" of the three functions that we consider here is the Q-function (or Husimi function). The original definition of the Q-function was originally [Hu] given as

$$Q(\alpha) = \frac{\langle \alpha | \hat{\rho} | \alpha \rangle}{\pi} \qquad \left(0 < Q(\alpha) < \frac{1}{\pi} \right)$$

(see D.6 for a proof).

The Q function, analogously to what happens to the Wigner function, can be expressed as a Gaussian convolution of the P function

$$Q(\alpha) = \frac{1}{\pi} \int d^2\beta \; P(\beta) \exp\left[-|\alpha - \beta|^2 \right]$$

(the proof being analogous to the one for the Wigner function).

Unsurprisingly, the Q function is convenient for evaluating anti-normally ordered correlators

$$\langle a^m a^{\dagger n} \rangle = \int d^2\alpha \; \alpha^m \alpha^{* n} \, Q(\alpha, \alpha^*)$$

Examples of Q functions

1. Coherent state $|\alpha_0\rangle$

$$Q(\alpha) = \frac{|\langle\alpha|\alpha_0\rangle|^2}{\pi} = \frac{\exp[-|\alpha-\alpha_0|^2]}{\pi}$$

2. Number state $|n\rangle$

$$Q(\alpha) = \frac{|\langle\alpha|n\rangle|^2}{\pi} = \frac{|\alpha|^{2n}\exp[-|\alpha|^2]}{\pi\, n!}$$

3. Squeezed state $|\alpha, r\rangle$ $\qquad X_1^\alpha = 2\,\mathrm{Re}\,\alpha\,,\ X_2^\alpha = 2\,\mathrm{Im}\,\alpha$

$$Q(x_1, x_2) = \frac{1}{4\pi^2\,\mathrm{ch}\, r}\exp\Big[-\frac{1}{2}\Big\{(x_1 - x_1^\alpha)^2\big(\exp[2r]+1\big) + (x_2 - x_2^\alpha)^2\big(\exp[-2r]+1\big)\Big\}\Big]$$

Appendix D

D1

Evaluating $\hat{A}$ in the position basis:

$$\langle x | \hat{A} | y \rangle = \frac{1}{(2\pi)^2} \int dq\, dp\, dk\, d\tau$$

$$a(q,p)\, \phi(k,\tau)\, \exp\left[i \frac{k\tau}{2}\right] \langle x | \exp\left[i k(\hat{q}-q)\right] \exp\left[i \tau(\hat{p}-p)\right] | y \rangle$$

$\hat{q}$ is diagonal in the position basis, while $\hat{p}$ is the generator of translations, therefore

$$\langle x | \hat{A} | y \rangle = \frac{1}{(2\pi)^2} \int dq\, dp\, dk\, d\tau \quad a(q,p)\, \phi(k,\tau)$$

$$\exp\left[i \frac{k\tau}{2}\right] \exp\left[i k(x-q)\right] \exp\left[-i \tau p\right] \langle x | y+\tau \rangle$$

writing $a(q,p)$ in terms of its Fourier transform

$$a(k,\tau) = \int dq\, dp \quad a(q,p) \quad \exp\left[-i k q\right] \exp\left[-i \tau p\right]$$

and considering the orthogonality of the position basis we have

$$\langle x | \hat{A} | y \rangle = \frac{1}{(2\pi)^2} \int dk\, d\tau\; a(k,\tau)\, \phi(k,\tau)\; \exp\left[i \frac{k\tau}{2}\right] \exp\left[i k x\right] \delta_{x,y+\tau}$$

with $\zeta = x + \frac{\tau}{2}$

$$\langle \zeta + \tfrac{\tau}{2} | \hat{A} | \zeta - \tfrac{\tau}{2} \rangle = \frac{1}{(2\pi)^2} \int dk \; a(k,\tau)\, \phi(k,\tau) \exp[ik\zeta]$$

and, inverting the FT

$$a(k,\tau)\,\phi(k,\tau) = 2\pi \int d\zeta \; \langle \zeta + \tfrac{\tau}{2} | \hat{A} | \zeta - \tfrac{\tau}{2} \rangle \exp[-ik\zeta]$$

which, in terms of $a(q,p)$, becomes

$$a(q,p) = \frac{1}{(2\pi)} \int dk \, d\tau \, d\zeta \; \frac{1}{\phi(k,\tau)} \exp[ikq] \exp[ip\tau] \exp[-ik\zeta] \, \langle \zeta + \tfrac{\tau}{2} | \hat{A} | \zeta - \tfrac{\tau}{2} \rangle$$

and if $\phi(k,\tau) = \phi_0(\tau)$ (i.e. indep of k)

$$a(q,p) = \int d\tau \; \frac{1}{\phi_0(\tau)} \exp[ip\tau] \, \langle q + \tfrac{\tau}{2} | \hat{A} | q - \tfrac{\tau}{2} \rangle$$

∎

$$d^2\eta = \begin{vmatrix} \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial \tau} \\ \frac{\partial \eta^*}{\partial x} & \frac{\partial \eta^*}{\partial \tau} \end{vmatrix} dx\, d\tau = \begin{vmatrix} -i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = -i\frac{1}{2} - i\frac{1}{2} = i$$

D.2

Defining $\eta = \frac{\tau}{\sqrt{2}} - i\frac{k}{\sqrt{2}}$, the characteristic function can be expressed as

$$\chi(\eta) = \mathrm{Tr}\left\{\hat{\rho}\, \exp\left[-ik\hat{q} - i\tau\hat{p}\right]\right\}$$

where, as usual, $\hat{q} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a})$, $\hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a})$.

Remembering the identity

$$\overset{-i\frac{k\tau}{2}}{\exp\left\{\frac{1}{2}[A,B]\right\}} = \overset{-ik\hat{q}}{\exp[A]}\,\overset{-i\tau\hat{p}}{\exp[B]}\,\overset{i(k\hat{q}+\tau\hat{p})}{\exp[-(A+B)]}$$

with $A = -ik\hat{q}$ and $B = -i\tau\hat{p}$ $\left([\hat{q},\hat{p}] = i\right)$, we can write

$$\exp\left[-ik\hat{q} - i\tau\hat{p}\right] = \exp\left[+i\frac{k\tau}{2}\right]\exp\left[-i\tau\hat{p}\right]\exp\left[-ik\hat{q}\right]$$

and therefore (cyclic property of the trace)

$$\chi(\eta) = \exp\left[i\frac{k\tau}{2}\right]\mathrm{Tr}\left\{\exp\left[-ik\hat{q}\right]\hat{\rho}\,\exp\left[-i\tau\hat{p}\right]\right\}$$

which, in the position basis, becomes

$$\chi(\eta) = \int dx\, \exp\left[i\frac{k\tau}{2}\right]\langle x|\exp\left[-ik\hat{q}\right]\hat{\rho}\,\exp\left[-i\tau\hat{p}\right]|x\rangle$$

which, taking into account the fact that $\hat{p}$ is the generator of translations, can be written as

$$\chi(\eta) = \int dx \, \exp\left[i\frac{k\tau}{2}\right] \exp\left[-ikx\right] \langle x | \hat{\rho} | x-\tau \rangle$$

The Wigner function $\left(\alpha = \frac{q}{\sqrt{2}} + i\frac{p}{\sqrt{2}}\right)$

$$W(q,p) = W(\alpha) = \frac{1}{\pi^2}\int d^2\eta \; \chi(\eta) \exp\left[\eta^*\alpha - \eta\alpha^*\right]$$

can thus be written as

$$W(q,p) =$$

$$\frac{1}{\pi^2}\int dk\,d\tau \int dx \, \exp\left[i\frac{k\tau}{2}\right] \exp\left[-ikx\right] \langle x | \hat{\rho} | x-\tau \rangle \cdot \exp\left[i(kq+\tau p)\right]$$

Implying $\phi_w = \frac{1}{2\pi}$ (see D.1). The previous expression can be further simplified to give

$$W(q,p) = \frac{1}{2\pi}\int d\tau \, \exp\left[ip\tau\right] \langle q+\tfrac{\tau}{2} | \hat{\rho} | q-\tfrac{\tau}{2} \rangle$$

With analogous steps it is possible to show that

$$P(\alpha) = P(q,p) =$$

$$\frac{1}{2\pi^2} \int d\kappa\, d\tau \int dx \; \exp\left[\frac{1}{4}\left(\tau^2 + \kappa^2\right)\right] \exp\left[i\frac{\kappa\tau}{2}\right]$$

$$\exp\left[-i\kappa x\right] \langle x | \hat{\rho} | x+\tau \rangle \cdot \exp\left[i\left(\kappa q + \tau p\right)\right]$$

$$\phi_P(\kappa,\tau) = \frac{1}{\pi} \exp\left[-\frac{1}{4}\left(\tau^2 + \kappa^2\right)\right]$$

and

$$Q(\alpha) = Q(q,p) =$$

$$\frac{1}{2\pi^2} \int d\kappa\, d\tau \int dx \; \exp\left[-\frac{1}{4}\left(\tau^2 + \kappa^2\right)\right] \exp\left[i\frac{\kappa\tau}{2}\right]$$

$$\exp\left[-i\kappa x\right] \langle x | \hat{\rho} | x+\tau \rangle \cdot \exp\left[i\left(\kappa q + \tau p\right)\right]$$

$$\phi_Q(\kappa,\tau) = \frac{1}{\pi} \exp\left[+\frac{1}{4}\left(\tau^2 + \kappa^2\right)\right]$$

$\blacksquare$

D3 Proof of $\rho = \int d^2\alpha \, P(\alpha) \, |\alpha\rangle\langle\alpha|$

If $\rho = \int d^2\alpha \, P(\alpha) \, |\alpha\rangle\langle\alpha|$, χ_N can be written as

D2.1 $$\chi_N(\eta) = \int d^2\alpha \, P(\alpha) \, \langle\alpha| e^{\eta a^\dagger} e^{-\eta^* a} |\alpha\rangle$$
$$= \int d^2\alpha \, P(\alpha) \, \exp\left[\eta\alpha^* - \alpha\eta^*\right]$$

Inverting the FT in D2.1, we get

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\eta \, \exp\left[\alpha\eta^* - \alpha^*\eta\right] \chi_N(\eta)$$

which is the definition of $P(\alpha)$ in terms of normally ordered characteristic function.

$\boxed{\text{M}}$

D.4 P-representation for a thermal state.

Combining 4.7 b and 4.6 a we obtain

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\eta \, \exp[\alpha\eta^* - \alpha^*\eta] \sum_n \bar{P}_n \langle n| e^{\eta a^\dagger} e^{-\eta^* a} |n\rangle$$

$$\bar{P}_n = \frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n$$

Writing the expectation value following anti normal-ordering, and inserting $\mathbb{1} = \frac{1}{\pi}\int d^2\alpha \, |\alpha\rangle\langle\alpha|$, we get

$$\langle n| e^{\eta a^\dagger} e^{-\eta^* a} |n\rangle = \frac{1}{\pi}\int d\alpha' \; e^{|\eta|^2} \langle n| e^{-\eta^* a} |\alpha\rangle\langle\alpha| e^{\eta a^\dagger} |n\rangle$$

one gets

$$\frac{1}{\pi^2}\int d^2\eta \; \exp[\alpha\eta^* - \alpha^*\eta] \; \frac{1}{\pi}\int d^2\alpha' \; e^{|\eta|^2} e^{\eta\alpha'^*} e^{-\eta^*\alpha'} \cdot$$

$$\exp[-|\alpha'|^2] \sum_n \bar{P}_n \frac{|\alpha'|^{2n}}{n!}$$

which, upon summation of the geometric series, gives

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\eta \; \exp[\alpha\eta^* - \alpha^*\eta]$$

$$\frac{1}{\pi}\int d^2\alpha' \; e^{|\eta|^2} \exp\left[\eta\alpha'^* - \eta^*\alpha\right] \frac{1}{\bar{n}+1} \exp\left[-|\alpha'|^2 + \frac{|\alpha'|^2}{1+\frac{1}{\bar{n}}}\right]$$

$$-|\alpha'|^2\left(1 - \frac{\bar{n}}{\bar{n}+1}\right)$$

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\eta \; \exp\left[\alpha\eta^* - \alpha^*\eta\right] \exp|\eta|^2$$

$$\frac{1}{\pi}\frac{1}{1+\bar{n}}\int d^2\alpha' \exp\left[\eta\alpha'^* - \eta^*\alpha'\right] \exp\left[-\frac{|\alpha'|^2}{\bar{n}+1}\right]$$

the α' integral can be solved by completing the square in the exponential.

$$\frac{1}{\pi}\int d\alpha \; \exp\left[-\lambda|\alpha|^2 + \nu\alpha^* + \nu^*\alpha\right] = \frac{1}{\lambda}\exp\left(-\frac{|\nu|^2}{\lambda}\right)$$

giving

$$\frac{1}{\pi}\frac{1}{1+\bar{n}}\int d^2\alpha' \; \exp\left[-\frac{\bar{n}}{\bar{n}+1}|\alpha'|^2 + \eta\alpha'^* - \eta^*\alpha'\right] = \frac{\bar{n}+1}{\bar{n}} \exp\left[-(\bar{n}+1)|\eta|^2\right]$$

and eventually

$$P(\alpha) = \frac{1}{\pi^2}\frac{1}{\bar{n}}\int d^2\eta \; \exp\left[\alpha\eta^* - \alpha^*\eta\right] \exp\left[-(n+1)|\eta|^2 + |\eta|^2\right]$$

$$= \frac{1}{\pi}\frac{1}{\bar{n}}\int d^2\eta \; \exp\left[-n|\eta|^2 + \alpha\eta^* - \alpha^*\eta\right] = \frac{1}{\pi\bar{n}}\exp\left[-\frac{|\alpha|^2}{\bar{n}}\right]$$

$\boxtimes$

D.5 Relation between the Wigner function and the P function

$$W(\alpha) = \frac{2}{\pi}\int d^2\beta \; P(\beta)\, \exp\left[-2\,|\beta-\alpha|^2\right]$$

$W(\alpha)$ can be written in terms of normal-ordered characteristic function as

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\eta \;\; \exp\left[-\frac{|\eta|^2}{2}\right] \chi_N(\eta)\, \exp\left[\eta^*\alpha - \eta\alpha^*\right]$$

which is the product of the FT of $P(\alpha)$ and a Gaussian (in Fourier space). We can thus express $W(\alpha)$ as the convolution product between $P(\alpha)$ and $FT^{-1}\left\{\exp\left[-\frac{|\eta|^2}{2}\right]\right\}$ which is given by

$$\frac{1}{\pi^2}\int d^2\eta \;\; \exp\left[-\frac{|\eta|^2}{2}\right] \exp\left[\eta^*\alpha - \eta\alpha^*\right] = \frac{2}{\pi}\, \exp\left[-2|\alpha|^2\right]$$

$\blacksquare$

D.6

$$Q(\alpha) = \frac{\langle \alpha | \rho | \alpha \rangle}{\pi}$$

From the definition of Q

$$Q(\alpha) = \frac{1}{\pi^2} \int d^2\eta \; \mathrm{Tr}\left\{ \rho \, e^{\eta(a-\alpha)} \, e^{-\eta^*(a^\dagger - \alpha^*)} \right\}$$

$$= \frac{1}{\pi^2} \int d^2\eta \, d^2\alpha' \, \langle \alpha' | \, \hat{\rho} \, e^{\eta(a-\alpha)} \, e^{-\eta^*(a^\dagger-\alpha^*)} \, | \alpha' \rangle$$

using the cyclic property of the trace

$$= \frac{1}{\pi} \int d^2\eta \, d^2\alpha' \, \langle \alpha' | e^{-\eta^*(a^\dagger-\alpha^*)} \, \hat{\rho} \, e^{\eta(a-\alpha)} | \alpha' \rangle$$

$$= \frac{1}{\pi^2} \int d^2\alpha' \underbrace{\int d^2\eta \, \exp\left\{ \eta\left(\alpha'^* - \alpha^*\right) - \eta^*(\alpha' - \alpha) \right\}}_{\pi \, \delta^{(2)}(\alpha'-\alpha)} \langle \alpha' | \hat{\rho} | \alpha' \rangle$$

$$= \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle$$

Covariance matrix

As anticipated above, the Gaussian character of W (or P and Q) hints that it is possible to extract information about correlations in the system under examination considering the functional form of the three representations considered. For a single-mode field, we can consider the following covariance matrix

$$C(a, a^\dagger) = \begin{bmatrix} \langle a^2 \rangle - \langle a \rangle^2 & \frac{1}{2} \langle a a^\dagger + a^\dagger a \rangle - \langle a^\dagger \rangle \langle a \rangle \\ \frac{1}{2} \langle a a^\dagger + a^\dagger a \rangle - \langle a^\dagger \rangle \langle a \rangle & \langle a^{\dagger 2} \rangle - \langle a^\dagger \rangle^2 \end{bmatrix}$$

which can also be rewritten in terms of correlation matrix as

$$C(x_1, x_2)_{p,q} = \frac{1}{2} \langle \hat{x}_p \hat{x}_q + \hat{x}_q \hat{x}_p \rangle - \langle \hat{x}_p \rangle \langle \hat{x}_q \rangle \qquad p,q = 1,2$$

which is related to $C(a, a^\dagger)$ by

$$C(\hat{x}_1, \hat{x}_2) = \Omega \, C(a, a^\dagger) \, \Omega^T$$

with

$$\Omega = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

N.B. in the appropriate basis $C(x_1, x_2)$ is diagonal, hence representing the "usual" position and momentum variances for the harmonic oscillator.

Crucially, $C(a, a^\dagger)$ can be related to the moments of α, α^* over P, W and Q respectively. In particular

$$\begin{aligned} C(a, a^\dagger) &= C_P(\alpha, \alpha^*) + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= C_W(\alpha, \alpha^*) \longrightarrow \text{calculate.} \\ &= C_Q(\alpha, \alpha^*) - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

(see 0.7 for a proof)

or, in terms of $x_1 = \alpha + \alpha^*$, $x_2 = i(\alpha^* - a)$

$$\begin{aligned} C(\hat{x}_1, \hat{x}_2) &= C_P(x_1, x_2) + I \\ & C_W(x_1, x_2) \\ & C_Q(x_1, x_2) - I \end{aligned}$$

– Verify $\sqrt{2}$

D.7

The (second-order) moments of α and α^* over $P(\alpha)$ can be written as

$$C_p(\alpha,\alpha^*)_{1,1} = \int d^2\alpha\, P(\alpha)\, \alpha^2 - \left[\int d^2\alpha\, P(\alpha)\, \alpha\right]^2$$

$$C_p(\alpha,\alpha^*)_{1,2} = \int d^2\alpha\, P(\alpha)\, |\alpha|^2 - \left|\int d^2\alpha\, P(\alpha)\, \alpha\right|^2$$

and analogously for $C(\alpha,\alpha^*)_{2,1}$ & $C(\alpha,\alpha^*)_{2,2}$.

The covariance matrix $C(a,a^\dagger)$ can be written through the P-representation of the density matrix $\hat{\rho} = \int d^2\alpha\, P(\alpha)\, |\alpha\rangle\langle\alpha|$

$$C(a,a^\dagger)_{1,1} = \int d^2\alpha\, P(\alpha)\, \langle\alpha|a^2|\alpha\rangle - \left[\int d^2\alpha\, P(\alpha)\, \langle\alpha|a|\alpha\rangle\right]^2$$

$$C(a,a^\dagger)_{1,2} = \int d^2\alpha\, P(\alpha)\, \frac{1}{2}\langle\alpha|(aa^\dagger + a^\dagger a)|\alpha\rangle -$$

$$\left|\int d^2\alpha\, P(\alpha)\, \langle\alpha|a|\alpha\rangle\right|^2$$

remembering that $\langle\alpha|\, aa^\dagger + a^\dagger a\, |\alpha\rangle = |\alpha|^2 + \frac{1}{2}$, proves the result for $C_p(\alpha,\alpha^*)$. The results for $W(\alpha)$, $Q(\alpha)$ can be proven through the convolution property. Ex.

4

Homodyne detection & quantum state tomography

What is the interest / power of the description in terms of the representations we have introduced here? It allows us, through measurement, to gain access to the state (Wigner function) of the system.

Let's focus here on a Gaussian state, and let's see how we can characterize its first two momenta (i.e. average and variance) and thus, in this case, completely characterize the state.

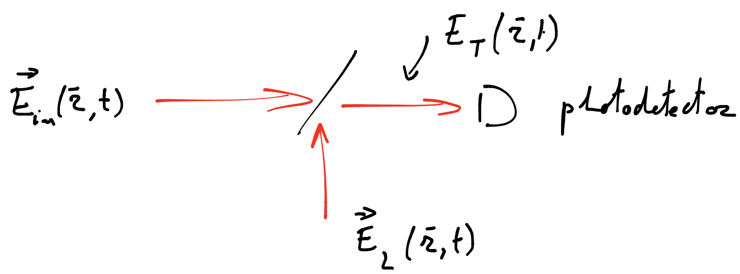
In order to achieve the result we want, we have to evaluate odd-ordered correlation functions,

$$G^{(n,m)}(x_1 \dots x_n, x_{n+1} \dots x_{n+m}) =$$
$$\langle E^-(x_1)\dots E^-(x_n)\, E^+(x_{n+1})\dots E^+(x_{n+m})\rangle$$

$G^{(n,m)}$ contains information about the phase of the state we are considering: in order to convince yourself about this consider $\langle E^+(x)\rangle$ for a coherent state.

The strategy consist in combining (homodyining) the incoming field with a local field whose phase is fixed, before the incoming field is detected.

The concrete example we consider here, consists of 2 fields of the same frequency which are combined through a beam splitter with transmittivity η.



As usual

$$E_{in}(\bar{r},t) = i\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{\frac{1}{2}}\left[a\, e^{i(\bar{k}\cdot\bar{r}-\omega t)} - a^{+} e^{-(\bar{k}\bar{r}-\omega t)}\right]$$

$$E_2(\bar{r},t) = i\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{\frac{1}{2}}\left[b\, e^{i(\bar{k}\cdot\bar{r}-\omega t)} - b^{+} e^{-(\bar{k}\bar{r}-\omega t)}\right]$$

After going through the beam splitter, the state of the field can be written as

$$E_T(\bar{r},t) = i\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{\frac{1}{2}}\left[c\, e^{i(\bar{k}\cdot\bar{r}-\omega t)} - c^{+} e^{-(\bar{k}\bar{r}-\omega t)}\right]$$

with

$$c = \sqrt{\eta}\, a + i\sqrt{1-\eta}\, b$$

The photon detector is sensitive to moments of $c^\dagger c$ (ideal detector theory). In particular it's possible to detect $M = c^\dagger c$.
We can calculate the mean photocurrent as

$$\langle c^\dagger c\rangle = \eta\langle a^\dagger a\rangle + (1-\eta) b^\dagger b \cdot i\sqrt{\eta(1-\eta)}\left(\langle a\rangle\langle b^\dagger\rangle - \langle a^\dagger\rangle\langle b\rangle\right)$$

since the local oscillator and the measured field are uncorrelated $\langle a^\dagger b\rangle = \langle a^\dagger\rangle\langle b\rangle$.
If the local oscillator (say E_2) is in a coherent state with large amplitude β, we have

$$\langle c^\dagger c\rangle \simeq (1-\eta)|\beta|^2 + |\beta|\sqrt{\eta(1-\eta)}\ \langle X_{\vartheta+\frac{\pi}{2}}\rangle$$

where

$$X_\vartheta = a e^{i\vartheta} + a^\dagger e^{i\vartheta} \qquad (\vartheta = \operatorname{Arg}\beta)$$

By changing ϑ it is thus possible to determine the amplitude of 2 canonically conjugated quadratures.
Analogously, it is possible to extract information about the fluctuation in the quadratures as

$$V(n) = (1-\eta)^2|\beta|^2 + |\beta|^2\eta(1-\eta)\ V(X_{\vartheta+\frac{\pi}{2}})$$

where the fluctuations of the local oscillator can be taken care of by balanced homodyne detection or by setting $\eta \gg 1$.

It is thus clear that, with the knowledge of the first two moments of the probability distribution as a function of θ it is possible to reconstruct the Wigner function of a Gaussian state. The reasoning can be generalized to higher moments for the reconstruction of Wigner functions which are not Gaussian.

5 Parametric effects

Degenerate parametric amplifier

In presence of a medium the electromagnetic field energy can be described (from the macroscopic point of view) on the same terms as it is in vacuum.

$$H = \int dV \, \frac{1}{2} \left(\varepsilon |\bar{E}|^2 + \mu |\bar{H}|^2 \right) \qquad \text{where } \bar{H} = \frac{\bar{B}}{\mu}$$

when, however, the medium "renormalizes" the permittivity ε and the permeability μ.

In general it is customary to rewrite $\varepsilon |\bar{E}|^2$ and $\mu |\bar{H}|^2$ as

$$\varepsilon_0 \varepsilon_r |\bar{E}| = \varepsilon_0 (\chi_e + 1) |\bar{E}|$$

and

$$\mu_0 \mu_r |\bar{H}| = \mu_0 (\chi_m + 1) |\bar{H}|$$

We can now focus on $|\bar{E}|$, but analogous results apply to $|\bar{H}|$.
For our purposes, we will consider here a class of media for which χ_0^e depends on $|\bar{E}|$. Examples of these materials are e.g. $TiBaO_3$, $GaSe$.
For weak enough nonlinearities, χ^e can be expanded in series of $|\bar{E}|$, giving for the polarization $|\bar{P}|$

$$\frac{P}{\varepsilon_0} = \chi^{(1)}|\bar{E}| + \chi^{(2)}|\bar{E}|^2 + \chi^{(3)}|\bar{E}|^3 + \dots$$

(in general $\chi^{(n)}$ is a tensor, but, for simplicity we neglect its tensor character here.

It is possible to show that, in presence of the appropriate second-order nonlinearity, the second-quantized expression for the Hamiltonian of the system can be written as:

$$H = \omega a^\dagger a + 2\omega c^\dagger c + g\left[c^\dagger a^2 + c\, a^{\dagger 2}\right]$$

where c and a are two modes whose frequency are given by 2ω and ω respectively and where the coupling g depends on the second-order susceptibility mediating the interaction between the modes.

If we assume that mode c (pump mode) is driven coherently with a strong pump r ($c \to r\exp[-i2\omega t]$), we have

$$H = \omega a^\dagger a + 2\omega|r|^2 +$$
$$g|r|\left[a^2\exp\{i(2\omega t+\varphi)\} + a^{\dagger 2}\exp\{-i(2\omega t+\varphi)\}\right]$$

which is the Hamiltonian for a degenerate parametric amplifier.

In the interaction picture

$$A_I = U^\dagger A U \qquad U = \exp[-iH_0 t]$$

this becomes

$$H_I = -i\frac{\chi}{2}\left(a^2 - a^{\dagger 2}\right)$$

and the Heisenberg EOMS are

$$\dot{a} = \chi a^\dagger \qquad \dot{a}^\dagger = \chi a$$

which can be solved to give

$$a(t) = a(0)\cosh\chi t + a^\dagger(0)\sinh\chi t$$

Light out of a parametric amplifier is thus supposed to be squeezed. This can be seen considering the two quadrature phase amplitudes

$$X_1 = a^\dagger + a \quad , \quad X_2 = i(a^\dagger - a)$$

for which the EOMs are given by

$$\dot{X}_1 = \chi X_1 \qquad \dot{X}_2 = -\chi X_2$$

This demonstrates how the parametric amplifier is a phase-sensitive amplifier, amplifying one quadrature and attenuating the other.

$$X_1(t) = e^{\chi t} X_1(0) \, , \qquad X_2(t) = e^{-\chi t} X(0)$$

For the variances we have that

$$V(X_1, t) = e^{2\chi t} V(X_1, 0) \qquad V(X_2, t) = e^{-2\chi t} V(X_2, 0)$$

For an initial vacuum or coherent state $V(X_i, 0) = 1$

$$V(X_1, 0) = e^{2\chi t} \qquad V(X_2, 0) = e^{-2\chi t}$$

with the variances satisfying the minimum uncertainty relation

$V(X_1)\,V(X_2) = 1$

The deamplified quadrature has less noise than the vacuum level, the amount of squeezing being proportional to the strength of the nonlinearity the amplitude of the pump and the interaction time.

If the system is initially in the vacuum state we have

$$g^{(2)}(0) = \frac{\langle a_t^+ a_t^+ a_t a_t \rangle}{\langle a_t^+ a_t \rangle^2} = \left(|S|^4 + |CS|^2\right) / |S|^4 = 1 + \frac{|C|^2}{|S|^2}$$

(see E1)

Meaning that the light generated from an initially vacuum state exhibits bunching.

Starting from a coherent state α (with $\alpha = |\alpha| e^{i\theta}$) one finds that

$$g^{(2)}(0) = 1 - \frac{S_2\left(\frac{1}{2} S_2 + 2 C_2\right)}{e^{-2\chi t}} < 1 \qquad \text{if} \quad \theta = \frac{\pi}{2}$$

$|\alpha| \gg \mathrm{sh}^2 \chi t,\ \mathrm{sh}\,\chi t,\ \mathrm{ch}\,\chi t$ E2

Under these conditions the light is antibunched.

The c.s. $|i|\alpha|\rangle$ evolves towards an amplitude squeezed state with a coherent amplitude $|\alpha| e^{-\chi t}$, this is related to the contraction along the X_2 direction.

Wigner function

In order to describe the full photon statistics, we evaluate the Wigner function for the degenerate parametric amplifier. To this aim, we evaluate the Wigner function.

We start from writing down the characteristic function

$$\chi(\eta,t) = \mathrm{Tr}\left\{ \rho(0)\, e^{\eta a_t^\dagger - \eta^* a} \right\}$$

If initially the system is in a coherent state we can write

$$\chi(\eta,t) = \exp\left[\eta\,\alpha_0^*(t) - \eta^*\alpha_0(t) - \frac{|\eta|^2}{2}\,\mathrm{ch}\,2\chi t + \frac{1}{4}\left(\eta^{*2}+\eta^2\right)\mathrm{sh}\,2\chi t \right]$$

with

$$\alpha_0(t) = \alpha_0\,\mathrm{ch}\,\chi t + \alpha_0^*\,\mathrm{sh}\,\chi t$$

This result can be expressed as

$$\chi(\eta,t) = \exp\left[-\frac{|\eta|^2}{2}\,\mathrm{ch}\,2\chi t + \frac{1}{4}\left(\eta^2+\eta^{*2}\right)\mathrm{sh}\,2\chi t \right] \exp\left[\eta\,\alpha_0^*(t) - \eta^*\alpha_0(t) \right]$$

Leading to see E3

$$W(\alpha, t) = \frac{1}{2\pi} \exp\left[-\frac{2}{e_+}(q-q_0)^2 - \frac{2}{e_-}(p-p_0)^2\right]$$

so the Wigner function of a state undergoing parametric amplification is a 2D Gaussian with the variances given by the fluctuations in the quadratures q & p (X_1, X_2).

Non-degenerate parametric amplifier

Analogously to what can be obtained for the degenerate parametric amplifier, a $\chi^{(2)}$ nonlinearity can give rise to a nonlinear interaction which, in this case, can be described by the Hamiltonian

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + i\chi\left(a_1^\dagger a_2^\dagger e^{-2i\omega t} - a_1 a_2 e^{2i\omega t}\right)$$

The Heisenberg EOMs in the interaction picture can be written as

$$\dot{a}_1 = \chi a_2^\dagger$$

$$\dot{a}_2 = \chi a_1^\dagger$$

whose solution is given by

$$a_1(t) = a_1(0) \,\mathrm{ch}\, \chi + a_2^\dagger \,\mathrm{sh}\, \chi t$$

If the system is initially in a C.S. $|\alpha_1\rangle|\alpha_2\rangle$ the mean photon number is given by

$$\langle n_1(t)\rangle = \langle \alpha_1 \alpha_2 |\, a_1^\dagger(t)\, a_1(t) \,| \alpha_1 \alpha_2\rangle$$

$$= |\alpha_1 \,\mathrm{ch}\,\chi t + \alpha_2^* \,\mathrm{sh}\,\chi t|^2 + \mathrm{sh}^2 \chi t$$

The correlation between modes 1 & 2 exhibits interesting quantum features: as opposed to the case of the degenerate parametric amplifier, we can consider correlations between the two different modes and show how they exhibit genuinely quantum features.

Consider the moment $\langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle$. This can be expressed through the P-representation as

$$\langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle = \int d^2\alpha_1 \int d^2\alpha_2 \; |\alpha_1|^2 |\alpha_2|^2 \; P(\alpha_1, \alpha_2)$$

By means of the Hölder inequality,

$$\int |f g| d\mu < \left[\int |f|^2 d\mu \right]^{\frac{1}{2}} \left[\int |g|^2 d\mu \right]^{\frac{1}{2}}$$

we can write for positive $P(\alpha)$

$$\int d^2\alpha_1 \, d^2\alpha_2 \, |\alpha_1|^2 |\alpha_2|^2 \, P(\alpha_1, \alpha_2) \leq \left[\int d^2\alpha_1 \, d^2\alpha_2 \, |\alpha_1|^4 \, P(\alpha_1 \alpha_2) \right]^{\frac{1}{2}} \left[\int d^2\alpha_1 \, d^2\alpha_2 \, |\alpha_2|^4 \, P(\alpha_1 \alpha_2) \right]^{\frac{1}{2}}$$

Leading to the Cauchy-Schwarz inequality

$$\langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle \leq \left[\langle a_1^{\dagger 2} a_1^2 \rangle \langle a_2^{\dagger 2} a_2^2 \rangle \right]^{\frac{1}{2}}$$

which, for symmetric modes (as in our case here) can be written as

$$\langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle \leq \langle a_1^{\dagger 2} a_1^2 \rangle$$

which, in terms of the second order two-mode

$$g_{12}^{(2)}(0) = \frac{\langle a_1^{\dagger} a_1 a_2^{\dagger} a_2 \rangle}{\langle a_1^{\dagger} a_1 \rangle \langle a_2^{\dagger} a_2 \rangle}$$

and the 1-mode

$$g_i^{(2)}(0) = \frac{\langle a_i^{\dagger} a_i^{\dagger} a_i a_i \rangle}{\langle a_i^{\dagger} a_i \rangle^2}$$

correlation functions, can be written as

$$\left[g_{12}^{(2)}(0) \right]^2 \leq g_1^{(2)}(0) \; g_2^{(2)}(0)$$

or, for symmetrical systems

$$g_{12}^{(2)} \leq g_1^{(2)}(0) \qquad \text{5.1}$$

However, we earlier (see "Properties of correlation functions") derived a relation among correlation functions, which can be written as

$$\langle a_1^{\dagger} a_1 a_2^{\dagger} a_2 \rangle^2 \leq \langle (a_1^{\dagger} a_1)^2 \rangle \langle (a_2^{\dagger} a_2)^2 \rangle$$

or, again for symmetrical systems,

$$\langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle \leq \langle a_1^{\dagger 2} a_1^2 \rangle + \langle a_1^\dagger a_1 \rangle$$

which can be written as

$$g_{12}^{(2)}(0) \leqslant g_1^{(2)}(0) + \frac{1}{\langle a_1^\dagger a_1 \rangle} \qquad 5.2$$

We will show now that a state initially prepared in the vacuum state leads to the violation of inequality 5.1 and to the "saturation" of the inequality 5.2.

It is possible to show that for the parametric amplifier the following conservation law holds

$$n_1(t) - n_2(t) = n_1(0) - n_2(0)$$

$n_i(t) = a_i^\dagger(t)\, a_i(t)$.

With this conservation law the intensity correlation can be written as

$$\langle n_1(t)\, n_2(t) \rangle = \langle n_1(t)^2 \rangle + \langle n_1(t) \left[n_2(0) - n_1(0) \right] \rangle$$

If the system is initially in the vacuum state the left term is zero, leading to

$$\langle n_1(t)\, n_2(t)\rangle = \langle a_1^\dagger(t)\, a_1^\dagger(t)\, a_1(t)\, a_1(t)\rangle + \langle a_1^\dagger(t)\, a_1(t)\rangle$$

which corresponds to the maximum violation of the Cauchy-Schwarz inequality 5.1.

This system thus exhibits quantum mechanical correlations which violate certain classical inequalities, this will be further discussed when talking about 2-mode squeezing and EPR paradox in this context.

E1

Exp. value of n on the sq. state $\zeta = \chi t$

$$\langle n \rangle = \langle 0| S^\dagger(\zeta)\, a^\dagger a\, S(\zeta)|0\rangle = |s|^2$$

$$S^\dagger(\zeta)\, a\, S(\zeta) = a \cosh|\zeta| + a^\dagger e^{i\varphi} \sinh|\zeta|$$

$$\langle a^\dagger a^\dagger a a \rangle =$$

$$= \langle 0| S^\dagger(\zeta)\, a^\dagger a^\dagger a a\, S(\zeta)|0\rangle = \langle 0| (c\, a^\dagger + s^* a)^2 (c a + s a^\dagger)^2 |0\rangle =$$

$$\langle 0| \left[c^2 \cancel{a^{\dagger 2}} + s^{*2} a^2 + cs(\cancel{2 a^\dagger a} + 1) \right] \left[c^2 \cancel{a^2} + s^2 a^{\dagger 2} + cs(\cancel{2 a^\dagger a} + 1) \right] |0\rangle$$

$$= |s|^4 + |c|^2 |s|^2$$

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} = 1 + \frac{\cosh^2 \chi t}{\sinh^2 \chi t}$$

E2

$$\langle 0| D^\dagger(\alpha)\, S^\dagger(\zeta)\, n\, S(\zeta)\, D(\alpha)\, |0\rangle$$

$$\langle \alpha | S^\dagger(\zeta)\, n\, S(\zeta) |\alpha\rangle = \langle \alpha | (a c + a^\dagger s)(a^\dagger c + a s^*) |\alpha\rangle$$

$$= \langle \alpha | \left(a a^\dagger c^2 + a^2 s^* c + a^{\dagger 2} s c + a^\dagger a |s|^2 \right) |\alpha\rangle$$

$$= (\alpha^2 s^* c + \alpha^{*2} s c) + |\alpha|^2 |s|^2 + |\alpha|^2 (c^2 + 1)$$

$$= |\alpha|^2 (\cosh 2\zeta + \cos 2\theta \sinh 2\zeta) + |\alpha|^2 \overset{\theta \to \frac{\pi}{2}}{\simeq} |\alpha|^2 \exp[-\chi t]$$

$$\langle\alpha| S^\dagger(\xi)\, a^\dagger a^\dagger a a\, S(\xi)|\alpha\rangle = \langle\alpha| (a c + a^\dagger s)^2 (a^\dagger c + a s)^2 |\alpha\rangle =$$

$$= \langle\alpha| \left[a^2 c^2 + a^{\dagger 2} s^2 + (a a^\dagger + a^\dagger a) c s\right]\left[a^{\dagger 2} c^2 + a^2 s^2 + (a^\dagger a + a a^\dagger) c s\right]|\alpha\rangle$$

$$= \langle\alpha| \left[a^2 a^{\dagger 2} c^4 + a^4 s^2 c^2 + a^2 (2 a^\dagger a + 1) c^3 s + \right.$$
$$a^{\dagger 4} s^2 c^2 + a^{\dagger 2} a^2 s^4 + a^{\dagger 2} (2 a^\dagger a + 1) c s^3 +$$
$$(2 a^\dagger a + 1) a^{\dagger 2} c^3 s + (2 a^\dagger a + 1) a^2 c s^3 + (2 a^\dagger a + 1)^2 c^2 s^2 |\alpha\rangle$$

$$= \langle\alpha| \left[(a^\dagger a)^2 - 2 a a^\dagger\right] c^4 + a^4 s^2 c^2 + (2 a^\dagger a + 5) a^2 c^3 s +$$
$$a^{\dagger 4} s^2 c^2 + a^{\dagger 2} a^2 s^4 + a^{\dagger 2} (2 a^\dagger a + 1) c s^3 +$$
$$a^{\dagger 2} (2 a^\dagger a + 5) c^3 s + (2 a^\dagger a + 1) a^2 c s^3 + (2 a^\dagger a + 1)^2 c^2 s^2 |\alpha\rangle$$

$$= \langle\alpha| (a^\dagger a^\dagger a a + a^\dagger a - 2 a^\dagger a - 2)\, c^4 + a^4 s^2 c^2 + (2 a^\dagger a + 5) a^2 c^3 s +$$
$$a^{\dagger 4} s^2 c^2 + a^{\dagger 2} a^2 s^4 + a^{\dagger 2} (2 a^\dagger a + 1) c s^3 +$$
$$a^{\dagger 2} (2 a^\dagger a + 5) c^3 s + (2 a^\dagger a + 1) a^2 c s^3 + (2 a^\dagger a + 1)^2 c^2 s^2 |\alpha\rangle$$

for $\vartheta = \frac{\pi}{2}$ $\quad (\vartheta = \operatorname{Arg}\alpha)$

$$= (|\alpha|^4 - |\alpha|^2 - 2) |c|^4 + \cancel{\alpha^4 s^2 c^2} + \cancel{(2|\alpha|^2 + 5)\alpha^2 s c^3} + \cancel{\alpha^{*4} s^2 c^2} + |\alpha|^4 s^4$$
$$+ \cancel{(2|\alpha|^2 + 1)\alpha^{*2} c s^3} + \cancel{(2|\alpha|^2 + 5) s c^3 \alpha^{*2}} + \cancel{(2|\alpha|^2 + 1)\alpha^2 c s^3} + (4|\alpha|^4 + 8|\alpha|^2 + 1) c^2 s^2 =$$

$$|\alpha|^4 c^4 + |\alpha|^4 s^4 + (4|\alpha|^4 + 8|\alpha|^2 + 1) c^2 s^2 - |\alpha|^2 c^4 - 2 c^4 =$$

$$\left(|\alpha|^4 - |\alpha|^2 - 2\right) c^4 + |\alpha|^4 s^4 + \left(4|\alpha|^4 + 8|\alpha|^2 + 1\right) s^2 c^2$$

$$|\alpha|^4 c^4 + |\alpha|^4 s^4 + 4|\alpha|^4 s^2 c^2 = |\alpha|^4 (c^4 + s^4) + |\alpha|^4 s_2^2$$

$$|\alpha|^4 (c^4 + s^4 + 2 s^2 c^2) - 2|\alpha|^4 s^2 c^2 + |\alpha^4| s_2^2$$

$$|\alpha|^4 \left(c_2^2 + s_2^2\right) - 2|\alpha|^4 s^2 c^2$$

$$|\alpha^4| \left(c_2^2 + s_2^2 - 2 s_2 c_2\right) - 2|\alpha|^4 s^2 c^2 + 2 s_2 c_2 |\alpha|^4$$

$$|\alpha|^4 (c_2 - s_2)^2 - |\alpha|^4 s_2 \left(\frac{1}{2} s_2 + 2 c_2\right)$$

$$V(n) = |\alpha|^4 (c_2 - s_2)^2 - |\alpha|^4 s_2 \left(\frac{1}{2} s_2 + 2 c_2\right)$$

$$\frac{V(n)}{\bar{n}} = 1 - \frac{s_2\left(\frac{1}{2} s_2 + 2 c_2\right)}{e^{-2\chi t}} < 1$$

$$[a^2, a^{\dagger 2}] = a[a, a^{\dagger 2}] + [a, a^{\dagger 2}]a = 2 a a^\dagger$$

$$[a^2, a^\dagger a] = a[a, a^\dagger a] + [a, a^\dagger a] a = 2 a^2 \qquad \left(a^2 a^\dagger a = a^\dagger a a^2 + 2a^2\right)$$

$$[a^\dagger a, a^{\dagger 2}] = a^\dagger [a^\dagger a, a^\dagger] + [a^\dagger a, a^\dagger] a^\dagger = 2 a^{\dagger 2}$$

E3 Upon replacing

$$\eta = \frac{1}{\sqrt{2}}\left(\tau + i\kappa\right) \qquad \eta^2 + \eta^{*2} = \left(\tau^2 - \kappa^2\right) \qquad |\eta|^2 = \frac{1}{2}\left(\tau^2 + \kappa^2\right)$$

The characteristic function can be written as

$$\chi(\eta,t) = \exp\left[-\frac{1}{4}\left(\tau^2+\kappa^2\right) \mathrm{ch}\, 2\chi t + \frac{1}{4}\left(\tau^2-\kappa^2\right) \mathrm{sh}\, \chi t\right]$$

$$= \exp\left[-\frac{\exp[-2\chi t]}{2}\tau^2 - \frac{\exp[2\chi t]}{2}\kappa^2\right] \exp\left[i\left(\kappa q_0 + \tau p_0\right)\right]$$

which, by F. transform

$$W(\alpha,t) = \frac{1}{\pi^2}\int d^2\eta \;\; \chi(\eta) \exp\left[\eta^*\alpha - \eta\alpha^*\right]$$

gives

$$W(\alpha,t) = \frac{1}{2\pi} \exp\left[-\frac{2}{e_+}\left(q-q_0\right)^2 - \frac{2}{e_-}\left(p-p_0\right)^2\right]$$

with $q = (\alpha^* + \alpha)/\sqrt{2}$, $p = i(\alpha^* - \alpha)/\sqrt{2}$, $e_\pm = \exp[\pm 2\chi t]$

$\blacksquare$

$| \; | \; (\; \} \quad \cos \qquad \}) \qquad | \; | \quad p[\quad]$