Open quantum systems

We will discuss here how to analyse the properties of quantum systems which are interacting with an environment. This analysis is relevant in the context of quantum optics on the one hand because the (unwanted) interaction between the system and the environment represents a detrimental effect that, to a large extent, cannot be avoided, and, on the other, it represents the "window" through which an experimenter has access to the system.

In the following we suppose that the Hamiltonian describing the system + the environment is given by

$$H_T = H_s + H_E + H_{sE}$$

where the three terms represent the Hamiltonians for the system (H_s), the environment (H_E) and their mutual interaction (H_{sE}).
Let's suppose that the environment can be described by a collection of Harmonic oscillators, i.e.

$$H_E = \sum_k w_k b_k^+ b_k$$

and that the coupling between system and environment can be written as

$$H_{sE} = i \sum_k g_k \left[b_k a^+ - a b_k^+ \right].$$

The Heisenberg EOMs for the environment can be written as

$$\dot{b}_k = -i\omega_k b_k + g_k a \quad (6.1)$$

which can be solved to give

$$b_{k,t} = b_{k,i} \; e^{-i\omega_k(t-t_i)} + g_k \int_{t_i}^{t} dt' \, a_{t'} \, e^{-i\omega_k(t-t')} \quad (6.2)$$

with $t > t_0$ (solution in terms of the input).
Eq. (6.1) can also be solved for $t < t_1$ (the output)

$$b_{k,t} = b_{k f} \; e^{-i\omega_k(t-t_f)} + g_k \int_{t}^{t_f} dt' \, a_{t'} \, e^{-i\omega_k(t-t')}$$

where $b_{k,i} = b_{k,t_i}$ and $b_{k,f} = b_{k,t_f}$
It is customary (but not necessary) to specify $t_i = -\infty$ and $t_f = +\infty$, in such a way that $b_k(t_0)$ and $b_k(t_1)$ are simply free fields.

Analogously to (6.1), it is possible to write the dynamics of the system field operator a as

$$\dot{a} = -i[H_s, a] - \Sigma_k \, g_k \, b_{k,t} . \quad (6.3)$$

Substituting (6.2) into (6.3), we get

$$\dot{a} = -i\left[H_s, a\right] - \sum_k g_k b_{k,i}\, e^{-i\omega_k(t-t_i)} - \sum_k g_k^2 \int_{t_i}^{t} dt'\, a_{t'}\, e^{-i\omega_k(t-t')} \qquad (6.4)$$

Let's assume that, in the range of frequencies of interest, each environment mode k is coupled with the same strength to the system $g_k^2 = \frac{\gamma}{2\pi D}$.

Moreover, we define the input field a_{in} as

$\hookrightarrow$ density of states

$$D = \frac{\partial k}{\partial \omega_k} = [T] = [\omega^{-1}] = [E]$$

$$a_{in}(t) = -\sqrt{\frac{1}{2\pi D}} \sum_k e^{-i\omega_k(t-t_i)}\, b_{ki}$$

$\hookrightarrow$ "sqrt of a flux"

with the equality

$$\frac{1}{2\pi D}\sum_k e^{-i\omega_k(t-t_i)} = \int \frac{d\omega}{2\pi} e^{-i\omega_k(t-t_i)} = \delta(t-t_i)$$

one can show that the input field satisfies

$$\left[a_{in\,t}, a^{\dagger}_{in\,t'}\right] = \delta(t-t') \qquad (6.5)$$

From (6.5) (expressing $\delta(t)$ as the limit of a continuous function)

$$\int_{t_i}^{t} dt'\, f(t')\,\delta(t-t') = \int_{t}^{t_f} dt'\, f(t')\,\delta(t-t') = \frac{1}{2} f(t),$$

and therefore (6.4) becomes

$$\dot{a} = i\left[H_s, a\right] - \frac{\gamma}{2} a + \sqrt{\gamma}\, a_{in} \qquad (6.6)$$

Eq. 6.6 is the Quantum Langevin equation for a system S (described by a field operator a), in contact with a bath (or environment) E.

The fact that 6.6 is an equation which is local in time crucially depends that the spectrum of the system/environment coupling is flat i.e. $g_k^2 = \frac{\gamma}{2\pi D}$ This approximation is referred to as Markov approximation, more in general, the Markov property refers to a stochastic process which is local in time (memoryless property).

Our goal is now to link the dynamics given by Eq. 6.6 to a description in terms of a master equation, describing the time evolution of the density matrix for an open quantum system.

In order to do this we have to introduce two tools/concepts: Itō process and superoperators in the Kraus representation.

An Itō process can be defined through the following property

$$dY(t) = D\,dt + W\,dB_t$$

where $\{B_t\}$ is the path of a brownian motion (nowhere differentiable).

It is possible to show that

$$(dY)^2 = w^2 dt$$

(while for a "regular" differential $(dY)^2 = 0$).

Ito calculus represents an extension of the usual rules of calculus to stochastic processes. In particular, the Riemann-Stieltjes integral is defined as

$$\int_a^b h(t)\, dG(t) \simeq \sum_j^n h(t_{j-1}) \left[G(t_j) - G(t_{j-1}) \right] \qquad 6.7$$

($G(t_i) = t_i$ representing the usual Riemann integral). The only requirement for the integral 6.7 to exist, is that the variation of G is finite, i.e.

$$V(G)[a,b] = \lim_{n \to \infty} \sum_{j=1}^{n} | G(t_j) - G(t_j) | < \infty$$

For a brownian motion $\{B_t\}$ this is not the case. However, it is possible to define

$$\int_0^t B_s\, dB_s \simeq \sum_k B(t_{k-1}) \left[B(t_k) - B(t_{k-1}) \right]$$

using the identity

$$B(t_{k-1})\left[B(t_k) - B(t_{k-1})\right] = \frac{1}{2}\left[B^2(t_k) - B^2(t_{k-1})\right] - \frac{1}{2}\left[B(t_k) - B(t_{k-1})\right]^2$$

we get

$$\sum_k B(t_{k-1})\left[B(t_k) - B(t_{k-1})\right] = \frac{1}{2}B^2(t) - \frac{1}{2}\sum_k\left[B(t_k) - B(t_{k-1})\right]^2$$

It turns out that (in probability)

$$\sum_k\left[B(t_k) - B(t_{k-1})\right]^2 = t$$

(for the central limit theorem, since $B(t_k) - B(t_{k-1})$ is normally distributed with $\mu = 0$ and $\sigma = t_n - t_{n-1}$) and therefore

$$\int_0^t B_s \, dB_s = \frac{1}{2}B^2(t) - \frac{1}{2}t$$

If $f(x) \in C^2$

$$df(B_t) = f'[B_t]\,dB_t + \frac{1}{2}f''[B_t]\,dt$$

By Taylor expansion

$$f(x+h) = f'(x)\ h + \frac{1}{2} h^2 f''(x) + R(x, h)$$

$$f(B_t) - f(0) = \sum_{k=0}^{n} \left[f(B_{t_k}) - f(B_{t_{k-1}}) \right]$$

$$= \sum_{k=0}^{n} f'(B_{t_{k-1}}) \left(B_{t_k} - B_{t_{k-1}} \right) + \frac{1}{2} \sum_{k=0}^{n} f''(B_{t_{k-1}}) \left[B_{t_k} - B_{t_{k-1}} \right]^2 + \varepsilon$$

for $n \to \infty$

$$f(B_t) - f(0) = \int_0^t f'(B_s)\, dB_s + \frac{1}{2} \int_0^t f''(B_s)\, ds$$

Kraus representation

Suppose that a bipartite $(S+E)$ system is initially characterized by a density matrix of the form

$$\rho = \rho_S \otimes |\psi_E\rangle\langle\psi_E|$$

The Schrödinger equation

$$\dot{\rho} = -i[H, \rho]$$

can be integrated to give

$$\rho_t = U_{SE}\, \rho_0\, U_{SE}^{+}$$

Suppose now we are interested in the system alone: the density matrix of S is obtained by partial trace over the environment E

$$\rho_S = Tr_E\left[U_{SE} (\rho_S \otimes |\psi_E\rangle\langle\psi_E|) U_{SE}^{+} \right]$$

$$= \sum_{\mu_E} \langle\mu_E| U_{SE} |\psi_E\rangle \rho_S \langle\psi_E| U_{SE} |\mu_E\rangle$$

where μ_E is a basis for the subsystem E (the environment).

The quantity $M_\mu = \langle \mu_E | U_{SE} | \psi_E \rangle$ represents an operator acting on subsystem S alone, through which the time evolution for ρ_S can be written as

$$\rho_S(t) = \$(\rho_{S0}) = \sum_\mu M_\mu \rho_{S0} M_\mu^+ \qquad (6.8)$$

From the unitarity of U_{SE} it follows that

$$\sum M_\mu^+ M_\mu = \mathbb{1}_S \qquad (6.9)$$

Regardless of its origin, if the linear map $\$$ exists and fulfills (6.9) it is called a superoperator (in the Kraus representation).
Why do we need this? We need it because we want to write an equation for the evolution of ρ_S which is local in time and depends on ρ_S alone, much in the same way as the QLE for a. Our goal now is thus to identify the operators M_μ from the QLEs which will allow us to write an equation for ρ_A: the master equation in the Lindblad form. N.B.: the Markov property of the Lindblad equation is directly "inherited" from the Markov property of the original QLE.

It is possible to show that the time evolution (within the Markov approximation) for the operator associated with the system is given by

$$a_t = \$ a = \sum_\mu M_\mu^+ a_0 M_\mu \qquad (6.10)$$

posing $t = dt$, and setting

$$M_0 = 1 + \hat{L}\, dB_s + K dt \,, \qquad M_\mu = O(dt^{3/2})$$

we can identify eq. (6.10) with the QLE eq. (6.6), since

$$a_{dt} = (1 + L^+ dB_s + K\, dt)\, a_0 (1 + L\, dB_s + K\, dt)$$

$$= a_0 + \{L^+ a_0 + a_0 L\} dB_s + [\{K, a_0\} + L^+ a_0 L]\, dt + O(dt^{3/2})$$

$("(dB_s)^2 = dt")$

From $\sum M_\mu^+ M_\mu = \mathbb{1}$, we have that $L^+ = L$ and $K = -\frac{1}{2} L^+ L$, leading to

$$a_{dt} = a_0 + [a_0, L]\, dB_s + [L^+ a_0 L - \tfrac{1}{2}\{L^+ L, a_0\}]\, dt \qquad (6.11)$$

posing

$$L = \sqrt{\kappa}\left(a_0^{+} a_{in} - a_0 a_{in}^{+}\right) dB_s$$

One can identify eq. 6.11 with 6.6.

Note:

Eq. 6.6 & 6.11 can be regarded as a "quantum" Itō process moving to the interaction picture, we can write 6.6 as

$$da = \hat{D}\, dt + \hat{W}\, dB_t$$

where $da = a_{dt} - a_0$, $\hat{D}\,dt$ represents the deterministic component of the time evolution and $\hat{W}\, dB_s$ the stochastic one.

Having identified the form of the Kraus operators, we can now write eq. 6.8 in terms of the operator L defined above, giving

$$\rho_{dt} = \rho_0 - 2\sqrt{\gamma}\, dt \left\{ [\rho_0, a^{+}]\, a_{in} - [\rho_0, a]\, a_{in}^{+} \right\}$$

$$+ dt \Big[\frac{\gamma}{2} a_{in} a_{in}^{+} \left(2 a \rho a^{+} - a^{+} a \rho - \rho a^{+} a\right) + \frac{\gamma}{2} a_{in}^{+} a_{in} \left(2 a^{+} \rho a - a a^{+} \rho - \rho a a^{+}\right)$$

$$+ \frac{\gamma}{2} a_{in} a_{in} \left(2 a^{+} \rho a^{+} - a a^{+} \rho - \rho a a^{+}\right) + \frac{\gamma}{2} a_{in}^{+} a_{in}^{+} \left(2 a \rho a - a a \rho - \rho a a\right) \Big]$$

6.11

The term $(2\sqrt{\gamma}t\ldots)$ is usually neglected (see Wells & Milburn). Moreover, if we assume the bath to be in thermal equilibrium the terms $\propto a_{in} a_{in}$ and $a^\dagger_{in} a^\dagger_{in}$ are 0 as well, leading to

$$\dot{\rho} = \frac{\gamma}{2}(N+1)\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right) + \frac{\gamma}{2}N\left(2a^\dagger\rho a - aa^\dagger\rho - \rho a a^\dagger\right) \quad (6.12)$$

From (6.11) (or (6.12)) it is possible to calculate the exp. value for system operators. Let's, for instance, evaluate $\langle n(t)\rangle$:

$$\frac{dn}{dt} = \mathrm{Tr}\{n\dot{\rho}\} = -\gamma\langle a^\dagger a\rangle + \frac{\gamma}{2}N$$

which gives

$$\langle n_t\rangle = \langle n_0\rangle \exp[-\gamma t] + N(1-e^{-\gamma t})$$

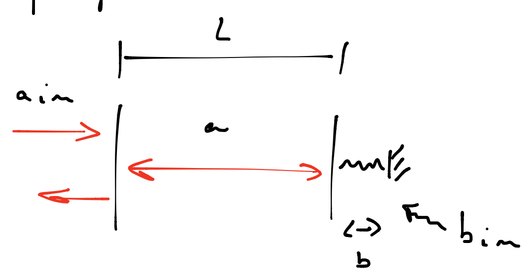
in the steady state $\langle n_t\rangle \to N$, which simply means that the system thermalises at the bath temperature, since

$$N(\omega_0) = \left[\exp\beta\omega - 1\right]^{-1}$$

Optomechanical systems

In recent years the possibility of manipulating mechanical degrees of freedom has been brought into the quantum regime in the context of optomechanical systems. Here we will discuss the dynamics of a (cavity) optomechanical system.

The system that we will consider is an optical cavity coupled to a mechanical resonator. An intuitive (but not so far from reality) picture is provided by 2 mirrors, one of which is semitransparent, and the other one is connected to a spring



The cavity is such that a single resonance can be resolved in the range of frequencies considered. More specifically, the condition for resonance is given by

$$n\frac{\lambda}{2} = L$$

which, in terms of resonant frequency, can be written as

$$\omega_c = \frac{2\pi}{\lambda} c = \frac{n\pi}{L} c \qquad (7.1)$$

and we assume that $\frac{\pi}{L} c$ is large enough for the single-resonance condition to hold.

As previously mentioned, one of the mirrors is allowed to move as if connected to a spring (one of the normal modes of a mechanical resonator), and therefore the total Hamiltonian of the system can be written as

$$H = \omega_c(x) a^\dagger a + \omega_m b^\dagger b \qquad (7.2)$$

where we can write

$$\omega_c(x) = \frac{n\pi}{L-x} \simeq \frac{n\pi}{L} - \frac{n\pi}{L^2} x = \omega_c + g x = \omega_c + g_0 (b^\dagger + b)$$

therefore from (7.2) we can write

$$H_{om} = \omega_c a^\dagger a + \omega_m b^\dagger b + g_0 a^\dagger a (b^\dagger + b) \qquad (7.3)$$

which is the canonical form of the optomechanical Hamiltonian.

Analogously to what we have been discussing in the parametric amplifier case, we will now consider the situation for which the system described by (7.3) is coherently driven by an external field. In this situation the QLEs for a and b associated with the Hamiltonian are given by

$$\dot{a} = -i\omega_c a - i g_0 a \left(b^\dagger + b \right) - \frac{\kappa}{2} a + \sqrt{\kappa}\, a_{in}$$

$$\dot{b} = -i\omega_m b - i g_0 a^\dagger a - \frac{\gamma}{2} b + \sqrt{\gamma}\, b_{in}$$

Since we are considering a situation for which a strong drive for the cavity field is present, we write the input field a_{in} as

$$a_{in} = \alpha_{in} e^{-i\omega_p t} + a$$

and consider the effects of a up to first order. With this assumptions, the Hamiltonian, in the interaction picture, can be written as

$$H_I = G a^\dagger \left\{ b^\dagger e^{i(\omega_m - \Delta)t} + b\, e^{-i(\omega_m + \Delta)t} \right\} + h.c. \qquad (7.4)$$

where $\Delta = \omega_p - \omega_c$, and $G = g_0 \alpha$. At this point, we can distinguish 2 interesting cases: one corresponds to the choice $\Delta = -\omega_m$

(red-detuned pump) and the other to $\Delta = \omega_m$ (blue-detuned pump).

For the red-detuned pump, 7.4 can be written as

$$H_I = G\, a^\dagger b + G\, a^\dagger b^\dagger e^{2i\omega_m t} + h.c. \qquad 7.5$$

We can now invoke the rotating-wave approximation, which, essentially, consists in neglecting terms which are oscillating (fast, at $2\omega_m$) and whose effect is assumed to be averaged out. With this approximation 7.5 becomes

$$H_I = G\left(a^\dagger b + a b^\dagger\right) \qquad 7.6$$

(beam-splitter Hamiltonian: $a_2 = \frac{1}{\sqrt{2}}\left(a_0 + i a_1\right)$, $a_3 = \frac{1}{\sqrt{2}}\left(i a_0 + a_1\right)$
$\hat{U} = \exp\left[i \frac{\pi}{2}\left(a_0^\dagger a_1 + a_1^\dagger a_0\right)\right]$)

The (linearized) QLEs associated with 7.6 can be written as

$$\dot{a} = -iGb - \frac{\kappa}{2}a + \sqrt{\kappa}\, a_{in}$$
$$\dot{b} = -iGa - \frac{\gamma}{2}b + \sqrt{\gamma}\, b_{in} \qquad (7.8)$$

going to Fourier space, (7.8) become

$$\chi_c^{-1} a + iGb = \sqrt{\kappa}\, a_{in}$$
$$iGa + \chi_m^{-1} b = \sqrt{\gamma}\, b_{in}$$

which can be solved to give

$$a = \chi_c \frac{\sqrt{\kappa}}{1+G^2\chi_c\chi_m} a_{in} - i\frac{G\chi_c\chi_m}{1+G^2\chi_c\chi_m}\sqrt{\gamma}\, b_{in}$$

$$b = \chi_m \frac{\sqrt{\gamma}}{1+G^2\chi_c\chi_m} b_{in} - i\frac{G\chi_c\chi_m}{1+G^2\chi_c\chi_m}\sqrt{\kappa}\, a_{in}$$

In this case b_{in} and a_{in} represent noise sources associated with the (finite) temperature of the photonic & phononic bath. The photonic bath can be safely considered to be at zero temperature and, therefore, the equation for b becomes

$$b = \frac{\sqrt{\gamma}}{\left(\frac{\gamma}{2} + \frac{2G^2}{\kappa}\right) - i\omega} b_{in}$$

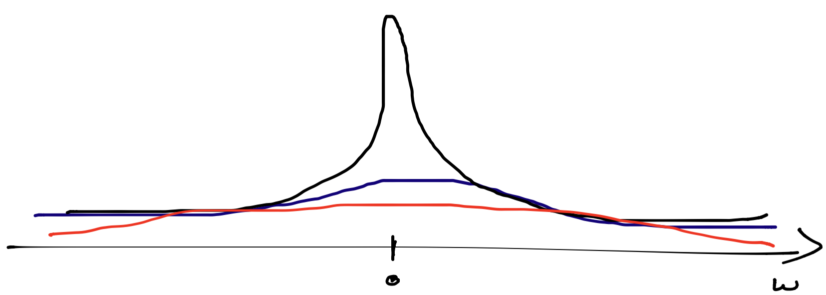
where, in addition, we have assumed that $\kappa \gg \gamma$.
One can then easily find

$$\langle n_\omega \rangle = \langle b^\dagger b \rangle = \frac{\gamma}{\frac{\gamma_e^2}{4} + \omega^2} \langle b^\dagger_{in} b_{in} \rangle$$

representing an (unnormalized) Lorentzian function, and

$$\gamma_e = \gamma + \frac{4G^2}{\kappa}$$

is the (optically induced) effective damping. If we plot n_ω for different values of G



It is thus somehow clear how by increasing $G = g_0 \alpha$, by increasing the drive α, the mechanical excitation spectrum gets depressed. The total number of excitations can be calculated as

$$\langle n_m \rangle = \int d\omega \, \langle n_\omega \rangle \overset{\kappa \gg \gamma}{\simeq} \int \frac{d\omega}{2\pi} \frac{\gamma_e}{\left(\frac{\gamma_e}{2}\right)^2 + \omega^2} \frac{\gamma}{\gamma_e} \simeq \frac{\gamma}{\gamma_e} n_b$$

which shows how, for large enough driving power the mechanical resonator can be put in its q.m. ground state.
A more refined calculation gives (see [Te])

$$n_m = n_b \left(\frac{\gamma}{\kappa} \frac{\kappa + 4G^2/\kappa}{\gamma + 4G^2/\kappa} \right) + n_c \frac{4G^2/\kappa}{4G^2/\kappa + \gamma_m}$$

If we now turn to the blue s.b. ($\Delta = \omega_m$) 7.4 becomes

$$H_I = G(a^+ b^+ + ab).$$

(parametric amplifier Hamiltonian)

An analogous calculation to the one leading to the cooling result, leads to

$$\chi_c^{-1} a + iG b^+ = \sqrt{\kappa}\, a_{in}$$
$$-iG a + \chi_m^{-1} b^+ = \sqrt{\gamma}\, b_{in}.$$

and thus

$$a = \chi_c \frac{\sqrt{\kappa}}{1 - G^2 \chi_c \chi_m} a_{in} - i \frac{G \chi_c \chi_m}{1 - G^2 \chi_c \chi_m} \sqrt{\gamma}\, b^{+}_{in}$$

$$b^{+} = \chi_m \frac{\sqrt{\gamma}}{1 - G^2 \chi_c \chi_m} b^{+}_{in} + i \frac{G \chi_c \chi_m}{1 - G^2 \chi_c \chi_m} \sqrt{\kappa}\, a_{in}$$

Neglecting b_{in}, the cavity field a, in presence of a small coherent drive $\delta\alpha_{in}$ becomes

$$a = \frac{\chi_m^{-1}}{(\chi_c \chi_m)^{-1} - G^2} \sqrt{\kappa}\, \delta\alpha_{in} \simeq \frac{2}{\kappa} \frac{\frac{\gamma}{2} - i\omega}{\frac{\gamma_e}{2} - i\omega} \delta\alpha_{in}$$

where $\gamma_e = \gamma - \frac{4G^2}{\kappa}$, resulting in amplification at $\omega = 0$ (in the rotating frame, see [Ma]).

What now if we have both pumps on? The Hamiltonian (with RWA) becomes

$$H_I = G_+ (a^+ b^+ + a b) + G_- (a^+ b + b^+ a)$$

If we define an operator $\hat{\beta}$ by

$$\hat{\beta} = b \, \mathrm{ch}\, z + b^{+} \mathrm{sh}\, z$$

with $\tanh z = \frac{G_+}{G_-}$, H_I can be written as

$$H_I = \mathcal{G}\left(a^{+}\beta + \beta^{+} a\right) \qquad 7.9$$

with $\mathcal{G} = \sqrt{G_-^2 - G_+^2}$.

We can therefore discuss the solution of 7.9 analogously to what is done for the ground-state cooling, but the state now achieved is a vacuum squeezed state (corresponding to the ground state for β).

In order to evaluate the squeezing effect of the mechanics, one has to evaluate the quadrature fluctuations which are given by

$$\langle |X^{\theta}|^2 \rangle = \frac{4\mathcal{G}^2 + \kappa^2\gamma}{4\mathcal{G}^2 + \gamma\kappa}(2n_m + 1) + \frac{4\left\{|G_+|^2 + |G_-|^2 - 2|G_+G_-|\cos 2\theta\right\}}{4\mathcal{G}^2 + \gamma\kappa}(2n_c + 1)$$

Quantum correlations and the EPR paradox

What is the argument of the paper by Einstein, Podolski & Rosen [Ei], in which they try to show that the description of reality as given by the wavefunction is not complete?

1) They introduce the idea of "element of reality". (ex. momentum eigenstate.

2) From the comm. relation it follows that either:

   a. Two noncomm. observables cannot have simultaneous reality,

   b. the description offered by Q.M. is not complete.

This is because the precise measurement of one of two noncommuting observables completely precludes the precise measurement of the other. Then, either they cannot be simultaneously real, or there must be a complete description at a "deeper" level, allowing for the exact determination of the two (noncommuting) observables.

3) Prove by contradiction that a. is false $\Rightarrow$ b. must be true.

How does the proof go?

Let's consider a state prepared leaving two subsystems interacting with one another. After the initial preparation the systems are brought far apart from each other. With a view to measuring quantity $\hat{A}$ on subsystem 1 we expand $\psi(x_1, x_2)$ in eigenstates of $u_n(x_1)$

$$\psi(x_1, x_2) = \sum_{\nu} \psi_n(x_2)\, u_n(x_1)$$

Now the measurement of $\hat{A}$ if the outcome a_n is obtained will project the wavefunction to

$$\psi(x_1 . x_2) \rightarrow \psi_n(x_2)\, u_n(x_1)$$

where we suppose that the initial preparation of the system is such that $\psi_n(x_2)$ is an eigenstate of an operator A'. Allowing us thus to state that the quantity associated with A' has an element of reality.

However, one could have equally chosen to measure a different quantity(B) on subsystem 1, and, analogously, the initial preparation could have been such that the state of system 2 would be in an eigenstate of B' (not commuting with A'), ergo establishing that two non commuting operators A' B' can have an element of reality $\Rightarrow$ the description provided by Q.M. is not complete.

Where does this argument fail?

The point is that A', B' are not measured (or predicted) simultaneously $\rightarrow$ and this requirement would prevent them from reaching the conclusion that both A' and B' are simultaneously real. But they conclude:

"This makes the reality of P and Q [A', B' for us] depend upon processes of measurement carried out on the first system, which does not disturb the system in any way. No reasonable definition of reality could be expected to permit this."

How can we arrange such a setting in quantum optics? Let's consider a non-degenerate parametric amplifier and let's define 2 sets of canonically conjugated variables

$$X_i = a_i e^{i\theta} + a_i^{+} e^{-i\theta} \qquad i = 1,2 \qquad (5.3)$$

The variables X_i^{θ} and $X_i^{\theta+\frac{\pi}{2}}$ obey the commutation relation

$$\left[X_i^{\phi}, X_i^{\phi+\frac{\pi}{2}}\right] = -2i$$

and thus represent perfect candidates to discuss the argument in the EPR paper.

In order to measure the degree of correlation between the two modes we consider the quantity

$$V(\theta,\varphi) = \frac{1}{2} \langle (X_1^{\theta} - X_2^{\varphi})^2 \rangle .$$

If $V(\theta,\varphi) = 0$ then $\hat{X}_1^{\theta}$ and $\hat{X}_2^{\varphi}$ are perfectly correlated: measuring X_1^{θ} will give us the value of $\hat{X}_2$ with certainty.

How can we get this in a parametric amplifier?

The Hamiltonian

$$H_I = + i \chi \left(a_1^{\dagger} a_2^{\dagger} - a_1 a_2 \right)$$

can be written in terms of the quadratures operators in 5.3 as

$$H_I = -2\chi \sin(\theta+\varphi) \left(X_1^{\theta} X_2^{\varphi} - X_1^{\theta+\frac{\pi}{2}} X_2^{\phi+\frac{\pi}{2}} \right)$$
$$- 2\chi \cos(\theta+\varphi) \left(X_1^{\theta+\frac{\pi}{2}} X_2^{\phi} + X_1^{\theta} X_2^{\varphi+\frac{\pi}{2}} \right)$$

The EOM for X_1^θ is then

$$\dot{X}_1^\theta = -4\chi\left[X_2^\phi \cos(\theta+\varphi) - X_2^{\phi+\frac{\pi}{2}} \sin(\theta+\varphi)\right]$$

and if $\theta+\phi=0$ X_1^θ is coupled only to X_2^ϕ.

A word of caution here: the dynamics of this Hamiltonian describes the preparation of the system: the system is, after the initial preparation is subsequently left to evolve freely (e.g. the pump is switched off, or we're detecting photons on a distant screen).

A direct calculation allows us to find

$$V(\theta,\varphi) = \cosh 2\chi t - \sinh 2\chi t \cos(\theta+\varphi)$$

if $\theta+\varphi=0$

$$V(\theta,\varphi) = \exp[-2\chi t]$$

reflecting the fact that, if no interaction takes place the two modes are independent, while for long interaction times the correlation becomes perfect and therefore a measurement of X_1^θ leads to an increasingly certain value of X_2^φ. One could have also chosen to measure $X_1^{\theta-\frac{\pi}{2}}$, obtaining $X_2^{\theta+\frac{\pi}{2}}$.

☆

In the spirit of the EPR paper, therefore, we have determined that two noncommuting observables can concomitantly (and without direct measurement) have an "element of reality".
Up to this point, we have not proven wrong the idea according to which the description of reality provided by the wavefunction is incomplete, in the sense of the EPR paper: namely we haven't ruled out the possibility of the existence of a theory which, by encompassing "local hidden variables" allows for a complete description of reality, again in the sense of EPR paper.

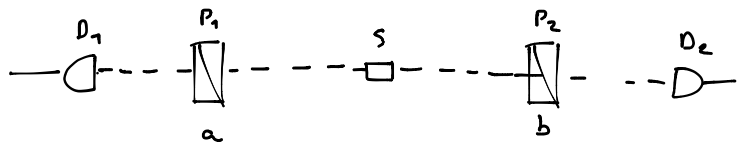
This question is solved by Bell's theorem (see [Be1]):

"No physical theory local hidden variables can ever reproduce all predictions of quantum mechanics."

This theorem is proven by showing how a theory of local hidden variables establishes inequalities which are violated in a quantum-mechanical setting. We will discuss here the CH (Clauser-Horne) inequality [Cl1]

CH: This is sometimes termed as "single-channel" Bell test.

The setup considered is the following



Pairs of particles (say photons) are directed towards two analysers, each consisting of an adjustable polariser and a detector.

The CH inequality establishes that

$$-1 \leq S \leq 0$$

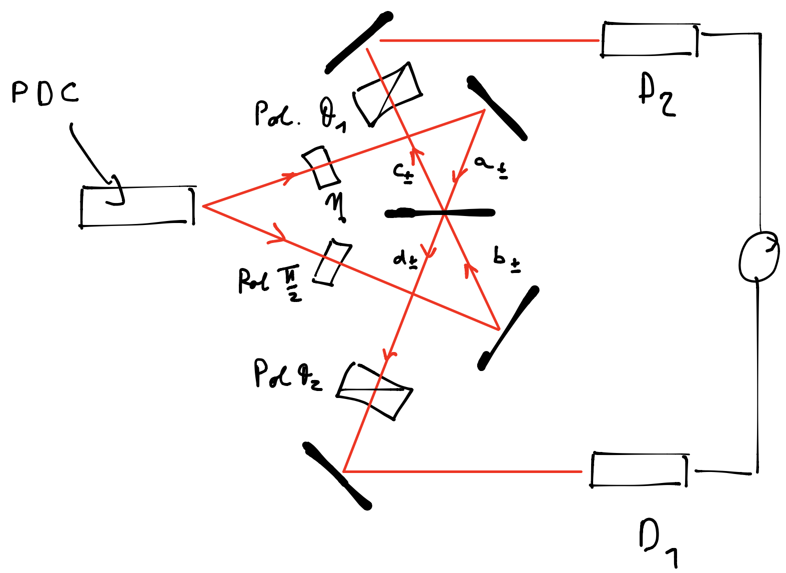
with

$$S = p_{12}(a,b) - p_{12}(a,b') + p_{12}(a',b) + p_{12}(a',b') - p_1(a',\infty) - p_2(\infty,b) \qquad (5.4)$$

(see E4 for a proof).

Violation of CH inequality in QO.

The following experiment was proposed by Reid & Walls [Re1] and the specific setup discussed here was realized by Ou and Mandel [Ou1]



☆

We consider two modes $a_\pm, b_\pm$ emitted by a nonlinear medium by parametric down conversion (along with their polarisation)

In the setup given above the modes $c_\pm, d_\pm$ will be given by

$$c_+ = \sqrt{T_+}\, b_+ + i\sqrt{R_+}\, a_+$$

$$c_- = \sqrt{T_-}\, b_- + i\sqrt{R_-}\, a_-$$

$$d_+ = \sqrt{T_+}\, a_+ + i\sqrt{R_+}\, b_+$$

$$d_- = \sqrt{T_-}\, a_- + i\sqrt{R_-}\, b_-$$

(we consider here the case in which an attenuator η is present for beam $a_\pm$ $\Rightarrow$ destroys "original" correlations due to the PDC)

$\hookrightarrow$ ex 13.2

$$a'_+ = \sqrt{\eta}\, a + \sqrt{1-\eta}\, v \quad (\Rightarrow \text{vacuum fluctuations})$$

The detected fields E_1^+ and E_2^+ are given by

$$\begin{cases} a_- = a_0 \operatorname{ch} \kappa + b_0^+ \operatorname{sh} \kappa \\ b_+ = b_0 \operatorname{ch} \kappa + a_0^+ \operatorname{sh} \kappa \end{cases}$$

$$(a_+^+ \cos\theta_1 + a_-^+ \sin\theta_1)(b_+^+ \cos\theta_2 + b_-^+ \sin\theta_2)$$

$$(b_+ \cos\theta_2 + b_- \sin\theta_2)(a_+ \cos\theta_1 + a_- \sin\theta_1)$$

$$< (a_0^+ \, c_\kappa s_1 + b_0 s_\kappa s_1)(b_0^+ c_\kappa c_2 + a_0 s_\kappa c_2)$$

$$(b_0 c_\kappa c_2 + a_0^+ s_\kappa c_2)(a_0 c_\kappa s_1 + b_0^+ s_\kappa s_1) >$$

$$c_\kappa^2 s_\kappa^2 s_1^2 c_2^2 \langle a_0^+ a_0 a_0 a_0^+ \rangle \qquad c_\kappa^4 s_1^2 c_2^2 \quad \langle a_0^+ b_0^+ b_0 a_0 \rangle$$

$$c_\kappa^2 s_\kappa^2 \; s_1^2 c_2^2 \langle b_0 b_0^+ b_0 b_0^+ \rangle^+ \qquad s_\kappa^2 c_\kappa^2 s_1^2 c_2^2 \quad \langle a_0^+ a_0 b_0 b_0^+ \rangle^+$$

$$s_\kappa^2 c_\kappa^2 s_1^2 c_2^2 \quad \langle b_0 b_0^+ a_0^+ a_0 \rangle$$

$$s_\kappa^4 \; s_1^2 c_2^2 \quad \langle b_0 a_0 a_0^+ b_0^+ \rangle$$

$$E_1^+ = c_+ \cos\theta_1 + c_- \sin\theta_1$$
$$E_2^+ = d_+ \cos\theta_2 + d_- \sin\theta_2$$

The two-photon detection probability (for perfect detector efficiency)

$$P(\theta_1, \theta_2) = \langle \psi | E_1^- E_2^- E_2^+ E_1^+ | \psi \rangle$$

If the PDC has low efficiency, we can assume that the state $|\psi\rangle$ is a two-photon state $a_-^+ b_+^+ |0\rangle = |1,1\rangle$, yielding

$$P(\theta_1, \theta_2) = \eta_0 \left(\sqrt{R_+ R_-} \sin\theta_1 \cos\theta_2 + \sqrt{T_+ T_-} \cos\theta_1 \sin\theta_2\right)^2$$

For a 50/50 beam splitter

$$P(\theta_1, \theta_2) = \frac{1}{4} \eta_0 \sin^2(\theta_1 + \theta_2) \qquad 5.5$$

With a view to investigate the CH inequality, we must also consider what happens in absence of one of the polarisers, i.e. we have to evaluate

$$P(\infty, \theta_2) = \sum_i \langle \psi | c_i^+ E_2^- c_i E_2^+ | \psi \rangle$$

Leading to (for a 50/50 beam splitter)

$$P(\infty, \theta_2) = P(\infty, \theta_1) = \frac{1}{4}\eta \qquad 5.6$$

Substituting 5.5, 5.6 into 5.4 we obtain

$$S = \frac{1}{4}\eta \Big[\sin^2(\theta_1+\theta_2) - \sin^2(\theta_1+\theta_2') + \sin^2(\theta_1'+\theta_2') + \sin^2(\theta_1'+\theta_2) - 2 \Big]$$

For $\theta_1 = \frac{\pi}{8}$, $\theta_2 = \frac{\pi}{4}$, $\theta_1' = 3\frac{\pi}{8}$, $\theta_2' = 0$

$$S = \frac{1}{4}\eta\left(\sqrt{2}-1\right) > 0$$

which constitutes a violation of the CH inequality.

E4 Lemma 1

For x_1, x_2, y_1, y_2, X, Y

$0 \leq x_1 \leq X$, $0 \leq x_2 \leq X$, $0 \leq y_1 \leq Y$, $0 \leq y_2 \leq Y$

the following inequality holds

$$-XY \leq U \leq 0$$

where $U = x_1 y_1 - x_1 y_2 + x_2 y_1 + x_2 y_2 - Y x_2$ (see [CL1] for a proof)

Proof of 5.4 [CL1]

Let's define $p_1(\lambda, a)$ as the probability that detector 1 triggers a signal when D_1 has polarisation a, and λ represents the initial state of the system, for which no model is assumed. The only assumptions we will make in the following is that the detection process at detector 1 does not depend on the detection process at detector 2, and that the presence of the polariser cannot increase the probability of detection: "no enhancement" hypothesis.

If $p_i(\lambda, x)$ $\left(x = a, a', b, b' \quad i = 1,2 \right)$ is a bona-fide probability one has that

$$0 \leq p_i(\lambda, x) \leq 1$$

and thus one can apply Lemma 1 to establish the following inequality

$$-1 \leq p_1(\lambda,a)\, p_2(\lambda,b) - p_1(\lambda,a)\, p_2(\lambda,b') + p_1(\lambda,a')\, p_2(\lambda,b)$$
$$+ p_1(\lambda,a')\, p_2(\lambda,b') - p_1(\lambda,a') - p_2(\lambda,b) \leq 0$$

Upon multiplication by the ensemble probability $\rho(\lambda)$ and integrating over λ, we get

$$-1 \leq p_{12}(a,b) - p_{12}(a,b') + p_{12}(a',b)$$
$$+ p_{12}(a',b') - p_1(a',\infty) - p_2(\infty,b) \leq 0$$

$\blacksquare$