FYSS6360 Plasma physics *Exercise 1/6* *(Monday 14 January 2019)*

1. Calculate with the Saha equation the ionization degree for the following two cases.
   (a) Ionospheric plasma with temperatures of 0.1 eV, 0.3 eV and 0.5 eV. Assume O^+ as the dominant ion species with 10^{11} cm^{-3} density.
   (b) Air at atmospheric pressure and 300 K temperature. Assume nitrogen as the dominant element.

2. Estimate the Debye length in the following situations. Find estimations for the information needed for the calculations from literature or online.
   (a) Flame
   (b) ECR ion source plasma ($n_e \approx 10^{11}$ cm^{-3} and $T_e \approx 100$ keV)
   (c) Core of a star

3. Estimate the fraction of the electrons which have energy in excess of 14.5 eV when the temperature corresponds to 2 eV and 5 eV.

   Hint 1: The Maxwell-Boltzmann distribution might be a good place to start.

   Hint 2: $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$

   $$\mathrm{erfc}(x) \approx \frac{e^{-x^2}}{\sqrt{\pi} x} \left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} + ...\right)$$

4. Consider a spherical plasma (radius R) consisting of singly charged ions and electrons with electron density n_e and electron temperature T_e. Due to thermal motion electrons near the surface can move outwards (leaving ions behind) by a distance Δs. The thickness of this sheath can be estimated by setting the potential energy of an electron at the surface to be equal with the average thermal energy $\langle E \rangle = \frac{3}{2} k T_e$ of the electrons. Estimate Δs. Express your answer in terms of the Debye length, λ_D.

5. Radiocommunication on Earth over long distances is based on application of MHz-range waves. The waves can be transmitted even without a direct line of sight between the transmitter and the receiver because of scattering from the ionospheric plasma. The MHz-range waves cannot propagate in the ionospheric plasma if the plasma oscillation frequency is greater than the frequency of the RF (we will see later during the course that this is the condition for wave propagation in plasma). Instead the waves are reflected and can thus reach the receiver. Even multiple "bounces" between the ionosphere and the ground are possible. Calculate the required plasma (electron) density in the ionosphere to inhibit propagation of waves within the range of $1 - 10$ MHz.

- Bonus question related to problem 5: Radio hobbyists can often experience trouble when transmitting/receiving during the night. Why is this? Drawing a sketch might be helpful.

Plasma physics, exercises 1 14.1.2019

1 (a) Dominant ion species O^+

$n_i(O^+) = 10^{11}\ 1/cm^3$, $T = 0{,}1\ eV / 0.3\ eV / 0.5\ eV$

Literature: $U = 13{,}62\ eV$

$k = 8.617 \cdot 10^{-5}\ eV/K$

Saha equation:

$$\frac{n_i}{n_n} = 3{,}0 \cdot 10^{27}\, T^{3/2}\, \frac{1}{n_i}\, e^{-U/T}$$

$[T] = eV$

$[n_i] = m^{-3}$

$$\frac{n_i}{n_n}(0{,}1 eV) = 3{,}0 \cdot 10^{27} \cdot (0{,}1\ eV)^{3/2} \cdot \frac{1}{10^{17}\ m^{-3}}\, e^{-13{,}62/0.1}$$

$$\approx \underline{6{,}7 \cdot 10^{-51}}$$

$$\frac{n_i}{n_n}(0{,}3 eV) \approx \underline{9{,}5 \cdot 10^{-11}} \qquad \frac{n_i}{n_n}(0{,}5 eV) \approx \underline{1{,}6 \cdot 10^{-2}}$$

(b) Literature: $U(N) = 14{,}53\ eV$, $1\,atm = 101325\ Pa$

Density of "air": $pV = NkT$

$$n_n = \frac{N}{V} = \frac{P}{kT}$$

$$= \frac{101325\ Pa}{1.38 \cdot 10^{-23}\ J/K \cdot 300K} \approx \underline{2.4 \cdot 10^{25}\ m^{-3}}$$

Multiplying Saha equation by $\frac{n_i}{n_n}$ gives

$$\left(\frac{n_i}{n_n}\right)^2 = 3.0 \cdot 10^{27}\, T^{3/2}\, \frac{1}{n_n}\, e^{-U/T}$$

$$\frac{n_i}{n_n} = \left[3.0 \cdot 10^{27} \cdot \big(\underbrace{300k \cdot 8.617 \cdot 10^{-5}\,\frac{eV}{K}}_{0.0259\ eV}\big)^{3/2} \cdot \frac{e^{-\frac{14.53}{0.0259}}}{2.4 \cdot 10^{25}\ m^{-3}}\right]^{1/2}$$

$$\approx \underline{\underline{10^{-122}}}$$

$[kT_e] = J$

(2) $\lambda_D = \sqrt{\dfrac{\varepsilon_0 k T_e}{n_e e^2}} = \sqrt{\dfrac{\varepsilon_0 k T_e}{n_e e}}$ where $[kT_e] = eV$

(a) Flame: $kT_e \approx 0{,}3\ eV$ $(= 3480\ K)$

$n_e \approx 10^{10}\ cm^{-3} = 10^{16}\ m^{-3}$

$$\Rightarrow \lambda_D = \sqrt{\frac{8{,}854 \times 10^{-12}\ C/Vm \cdot 0{,}3\ eV}{10^{16}\ m^{-3} \times 1{,}602 \times 10^{-19}\ C}}$$

$$\approx 4 \times 10^{-5}\ m = \underline{\underline{0.04\ mm}}$$

(b) ECRIS plasma: $kT_e = 100\ keV = 10^5\ eV$

$n_e = 10^{11}\ cm^{-3} = 10^{17}\ m^{-3}$

$$\Rightarrow \lambda_D = \sqrt{\frac{8{,}854 \times 10^{-12}\ C/Vm \cdot 10^5\ eV}{10^{17}\ m^{-3} \times 1{,}602 \times 10^{-19}\ C}}$$

$$\approx 7{,}4 \cdot 10^{-3}\ m = \underline{\underline{7{,}4\ mm}}$$

(c) Core of a star: $kT_e \approx 10^4\ eV$

$n_e \approx 10^{28}\ cm^{-3} = 10^{32}\ m^{-3}$

$$\Rightarrow \lambda_D = \sqrt{\frac{8{,}854 \times 10^{-12}\ C/Vm \cdot 10^4\ eV}{10^{32}\ m^{-3} \cdot 1{,}602 \cdot 10^{-19}\ C}}$$

$$\approx \underline{\underline{7{,}4 \cdot 10^{-11}\ m}}$$

(3) Idea: we want to integrate Maxwell-Boltzmann distribution from 14.5 eV to infinity. Let's do this with the velocity distribution:

$$p(v)\,dv = \underbrace{4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}}_{=A} v^2 e^{-\frac{mv^2}{2kT}} dv$$

$$= \underbrace{Av}_{u} \cdot \underbrace{v e^{-\frac{mv^2}{2kT}} dv}_{v'}$$

$$u' = A \qquad v = -\frac{kT}{m} e^{-\frac{mv^2}{2kT}}$$

Integration by parts:

$$\int_{v_1}^{\infty} uv' = \Big/_{v_1}^{\infty} uv - \int_{v_1}^{\infty} vu'$$

$$= \underbrace{\Big/_{v_1}^{\infty} -Av\frac{kT}{m} e^{-\frac{mv^2}{2kT}}}_{= 0 + Av_1\frac{kT}{m} e^{-\frac{mv_1^2}{2kT}} =: g(1)} + \underbrace{\int_{v_1}^{\infty} A\frac{kT}{m} e^{-\frac{mv^2}{2kT}} dv}_{(*)}$$

For integral (*), remember error function hint:

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2} du, \qquad \mathrm{erfc}(x) \approx \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 - \frac{1}{2x^2} + \dots\right)$$

The integral (*) has the same form, so we can turn it into erfc and use the above property by doing the following:

Let's define: $u^2 = \frac{mv^2}{2kT} \Rightarrow u = \sqrt{\frac{m}{2kT}}\, v$

$$\Rightarrow du = \sqrt{\frac{m}{2kT}}\, dv \Rightarrow dv = \sqrt{\frac{2kT}{m}}\, du$$

And we get

③ Continues...

$$\int_{v_1}^{\infty} A \frac{kT}{m} e^{-\frac{mv^2}{2kT}} \, dv$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} A \frac{kT}{m} \sqrt{\frac{2kT}{m}} \int_{\sqrt{\frac{m}{2kT}} v_1}^{\infty} e^{-u^2} \, du$$

erfc

$$= \frac{1}{\cancel{\sqrt{\pi}}} \frac{\cancel{\sqrt{\pi}}}{2} A \frac{kT}{m} \sqrt{\frac{2kT}{m}} \frac{e^{-\left(\sqrt{\frac{m}{2kT}} v_1\right)^2}}{\sqrt{\frac{m}{2kT}} v_1} \left(1 - \frac{1}{2\left(\sqrt{\frac{m}{2kT}} v_1\right)^2} + \dots\right)$$

$$= \underbrace{\sqrt{\frac{2kT}{\pi m}} \frac{1}{v_1} e^{-\left(\sqrt{\frac{m}{2kT}} v_1\right)^2} \left(1 - \frac{1}{2\left(\sqrt{\frac{m}{2kT}} v_1\right)^2} + \dots\right)}_{=: \, g(2)}$$

So finally we get

$$\int_{v_1}^{\infty} p(v) \, dv = g(1) + g(2)$$

And $v_1 = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2 \cdot 14.5 \text{ eV} \cdot 1{,}602 \cdot 10^{-19} \text{ J/eV}}{9{,}1 \cdot 10^{-31} \text{ kg}}}$

$\approx 2{,}26 \cdot 10^{6}$ m/s $\quad \left(2.2595 \cdot 10^{6} \frac{m}{s}\right)$

Also: $2 \text{ eV} = 23\,200 \text{ K}$

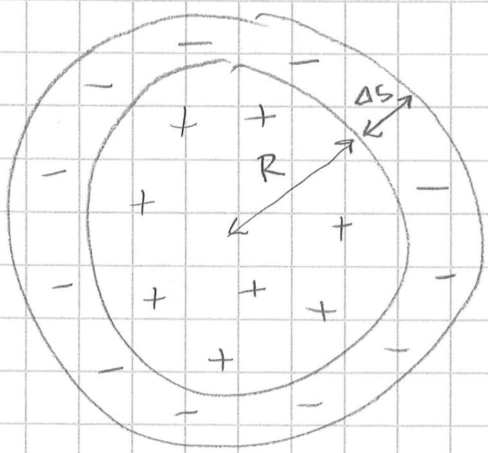
$5 \text{ eV} = 58\,000 \text{ K}$

Putting these numbers in the equations gives

(a) $2 \text{ eV} \Rightarrow 2{,}14 \cdot 10^{-3} + 1{,}37 \cdot 10^{-4} \approx 0{,}00228 \approx \underline{\underline{0{,}23\%}}$

(b) $5 \text{ eV} \Rightarrow 1{,}05 \cdot 10^{-1} + 1{,}50 \cdot 10^{-2} \approx 0{,}120 \approx \underline{\underline{12\%}}$

④ Sketch of situation:



Displacement of electrons causes a net positive charge build-up within R such that

$$Q_R = -Q_{R+\Delta S}$$

Because ΔS is small, we can estimate the number of electrons in the shell to be

$$N_e = 4\pi R^2 \cdot n_e \cdot \Delta S$$

and thus

$$Q_R = N_e \cdot e = 4\pi R^2 \cdot n_e \cdot \Delta S \cdot e$$

An electron at R experiences the electric attraction of Q_R, which imbues it with the electric potential energy (w.r.t. the electric potential φ)

$$U = e \cdot \varphi(R) = \frac{e \cdot Q_R}{4\pi\varepsilon_0 R} = \frac{e^2 \cancel{4\pi} R^{\cancel{2}} n_e \Delta S}{\cancel{4\pi}\varepsilon_0 \cancel{R}} = \frac{e^2 R n_e \Delta S}{\varepsilon_0}$$

Their thermal energy allows the electrons to scale the potential φ, i.e.

$$\frac{3}{2} k T_e = \frac{e^2 R n_e \Delta S}{\varepsilon_0}$$ from which we get

$$\underline{\underline{\Delta S}} = \frac{3}{2} \frac{k T_e \cdot \varepsilon_0}{e^2 n_e R} = \underline{\underline{\frac{3}{2R} \lambda_D^2}} \qquad \left(\lambda_D = \sqrt{\frac{\varepsilon_0 k T_e}{n_e e^2}} \right)$$

π Condition for inhibiting wave propagation:

$$\omega_{pe} > 10\text{ MHz} \cdot 2\pi$$

And $\omega_{pe} = \sqrt{\frac{e^2 n_e}{\varepsilon_0 m_e}}$ so we get

$$\sqrt{\frac{e^2 n_e}{\varepsilon_0 m_e}} > 10 \cdot 10^6 \cdot 2\pi \text{ Hz}$$

$$\Rightarrow n_{e,\text{limit}} = \frac{\varepsilon_0 m_e (10 \cdot 10^6 \cdot 2\pi \text{ Hz})^2}{e^2}$$

$$\approx \underline{\underline{1{,}24 \cdot 10^{12}\ 1/m^3}}$$

BONUS: During the night less photons interact with the ionosphere, leading to weakened photoionization and thus reduction of plasma density and consequently the plasma oscillation frequency decreases, too.

$\Rightarrow$ the limit frequency of RF that bounces back from ionosphere is reduced

$\Rightarrow$ the high-end frequencies that could be used during the day do not "bounce" back anymore during the night.