(1) Since

$$\begin{aligned}\mathbb{E}e^{\eta X} &= \int_0^\infty \mathbb{P}(e^{\eta X} > t)dt = \int_0^1 \mathbb{P}(\Omega)dt + \int_1^\infty \mathbb{P}(e^{\eta X} > t)dt \\ &= 1 + \int_0^\infty \mathbb{P}(e^{\eta X} > e^{\eta s})\eta e^{\eta s} ds = 1 + \eta \int_0^\infty \mathbb{P}(X > s) e^{\eta s} ds \\ &= 1 + \eta \int_0^\infty e^{\eta s}(1 - F(s))ds,\end{aligned}$$

we get $\int_0^\infty e^{\eta s}(1-F(s))ds = \frac{1}{\eta}\left(\mathbb{E}e^{\eta X} - 1\right)$.

(2) (a)

$$\frac{1-F(x)}{e^{-\lambda x}} = \frac{\left(\frac{b}{x}\right)^a}{e^{-\lambda x}} = b^a \frac{e^{\lambda x}}{x^a} \to \infty$$

as $x \to$ for any $\lambda > 0$, since the exponential function increases faster than any power function. Thus the distribution is heavy tailed.

(b)

$$\frac{1-F(x)}{e^{-\lambda x}} = \frac{e^{-cx^\tau}}{e^{-\lambda x}} = e^{\lambda x - cx^\tau} = e^{\lambda x(1 - \frac{c}{\lambda}x^{\tau-1})}$$

If $\tau < 1$, then $x^{\tau-1} \to 0$ as $x \to \infty$ and

$$\lim_{x\to\infty} e^{\lambda x(1-\frac{c}{\lambda}x^{\tau-1})} = \lim_{x\to\infty} e^{\lambda x} = \infty$$

for all $\lambda > 0$. Hence the Weibull distribution is heavy tailed for $\tau < 1$.
If $\tau \geq 1$, then $x^{\tau-1} \geq 1$ for all $x > 0$ and $-\frac{c}{\lambda}x^{\tau-1} \leq -\frac{c}{\lambda}$. Choose $\lambda < c$, so that $1 - \frac{c}{\lambda}x^{\tau-1} \leq 1 - \frac{c}{\lambda} < 0$. Then

$$\lim_{x\to\infty} e^{\lambda x(1-\frac{c}{\lambda}x^{\tau-1})} \leq \lim_{x\to\infty} e^{\lambda x(1-\frac{c}{\lambda})} = \lim_{x\to\infty} e^{(\lambda-c)x} = 0.$$

Hence the Weibull distribution is light tailed for $\tau \geq 1$.

(3) (a)

$$
\begin{aligned}
q_n &= \binom{m}{n} p^n (1-p)^{m-n} = \frac{m!}{(m-n)!n!} p^n (1-p)^{m-n} \\
&= \frac{(m-n+1)m!}{(m-n+1)(m-n)!(n-1)!n} p p^{n-1} \frac{1}{1-p} (1-p)^{m-n+1} \\
&= \frac{m-n+1}{n} \frac{m!}{(m-n+1)!(n-1)!} \frac{p}{1-p} p^{n-1} (1-p)^{m-n+1} \\
&= \left(-1 + \frac{m+1}{n}\right) \frac{p}{1-p} \frac{m!}{(m-(n-1))!(n-1)!} p^{n-1} (1-p)^{m-(n-1)} \\
&= \left(-\frac{p}{1-p} + \frac{(m+1)\frac{p}{1-p}}{n}\right) q_{n-1},
\end{aligned}
$$

so that $a = -\frac{p}{1-p}$ and $b = (m+1)\frac{p}{1-p}$.

(b)

$$
\begin{aligned}
q_n &= \binom{v+n-1}{n} p^v (1-p)^n = \frac{(v+n-1)!}{(v+n-1-n)!n!} p^v (1-p)^n \\
&= \frac{(v+n-2)!(v+n-1)}{(v-1)!(n-1)!n} p^v (1-p)(1-p)^{n-1} \\
&= \frac{v+n-1}{n} (1-p) \frac{(v+n-2)!}{(v-1)!(n-1)!} p^v (1-p)^{n-1} \\
&= \left(1 + \frac{v-1}{n}\right) (1-p) \frac{(v+(n-1)-1)!}{(v+(n-1)-1-(n-1))!(n-1)!} p^v (1-p)^{n-1} \\
&= \left(1 - p + \frac{(v-1)(1-p)}{n}\right) \binom{v+(n-1)-1}{n-1} p^v (1-p)^{n-1} \\
&= \left(1 - p + \frac{(v-1)(1-p)}{n}\right) q_{n-1},
\end{aligned}
$$

so that $a = 1-p$ and $b = (v-1)(1-p)$.

(4) $N(T,\omega) - N(T-,\omega) = 1$ if and only if there is a jump in the path at T, that means $T_n(\omega) = T$ for some $n = 1, 2, \ldots$. Therefore

$$
\begin{aligned}
\mathbb{P}(N(T) - N(T-) = 1) = \mathbb{P}(T_n = T \text{ for some } n = 1, 2, \ldots) &= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{T_n = T\}\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(T_n = T).
\end{aligned}
$$

We have $T_n = W_1 + \cdots + W_n$, where $W_1, \ldots, W_n$ are independent and have exponential distribution. Then also T_n has a density function h_n (Exercises -1-) and

$$
\mathbb{P}(T_n = T) = \int_{\mathbb{R}} \mathbb{1}_{\{T\}}(x) h_n(x) dx = \int_T^T h_n(x) dx = 0.
$$

Hence $\mathbb{P}(N(T) - N(T-) = 1) = 0$.