Gauge invariance and superconduc

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U(1) symmetry breaking and superconducting current

Let's suppose that we multiply all wave functions by the same phase factor $e^{i\varphi}$. Then field operators transform like $\psi \to e^{i\varphi}\psi$. It is easy to see that the normal propagators are invariant with respect to this global U(1) transformation

$G = \langle T\psi\psi^+\rangle$. On the other hand the anomalous GF is not

$F = \langle T\psi\psi\rangle \to e^{2i\varphi}\langle T\psi\psi\rangle$. This corresponds to the physical origin of superconducting phase transition which breaks the U(1) symmetry. It is less symmetric than normal state. The phase shift requires no energy – therefore it correspond to the Goldstone mode.

What if we consider the local U(1) symmetry, that is assume that the phase depends on the coordinate. We should take into account the gauge field – vector potential because the Goldstone mode can be eaten by gauge bosons, which becomes massive.

In superconductors it results in Meissner screening the repulsion of electromagnetic field from superconducting state

To see how that happens let us consider the gauge-invariant Gorkov equations:

$$\begin{cases} \left(-i\omega_n + \frac{\hat{p}_1^2}{2m} - \mu\right) G + \Delta F^+ = \delta(r_1 - r_2) \\ \left(-i\omega_n - \frac{\hat{p}_1^{*2}}{2m} - \mu\right) F^+ + \Delta^* G = 0 \end{cases}$$

Here $\hat{p} = -i\nabla - \frac{e}{\hbar c}A$, $\hbar, c = 1$

The gap function Δ can have space dependent phase $\Delta = |\Delta| e^{2i\varphi(r)}$

Let's eliminate it by local gauge transformation

$$G(r_1, r_2) = e^{-i\frac{\varphi(r_1)}{2} + i\frac{\varphi(r_2)}{2}}\, \tilde{G}(r_1, r_2)$$

$$F^+(r_1, r_2) = e^{+i\frac{\varphi(r_1)}{2} + i\frac{\varphi(r_2)}{2}}\, \tilde{F}^+(r_1, r_2)$$

Then for the new functions we get

$$\begin{cases} \left(-i\omega_n + \frac{(p_1 + mv_s)^2}{2m} - \mu\right) G + |\Delta| F^+ = \delta(r_1 - r_2) \\ \left(-i\omega_n - \frac{(p_1^* - mv_s)^2}{2m} - \mu\right) F^+ + |\Delta| G = 0 \end{cases}$$

Here $p_1 = -i\nabla_1$

$2mv_s = \left(\nabla\varphi - \frac{2e}{\hbar c}A\right)$

condensate (Cooper pair) velocity

Assuming that $|\Delta|$ is space-independent we can use momentum repr.
$G(r_1 r_2) = G e^{ip(r_1-r_2)}$

If velocity is small, then

$$\frac{(p+mV_s)^2}{2m} - \mu = \xi_p + \bar{p}\bar{V}_s \quad \text{- the Doppler shift of energy}$$

Hence in Gorkov Eqs. we can just substitute

$$\omega_n \to \tilde{\omega}_n + i\bar{p}\bar{V}_s$$

Current

In order to find how electromagnetic field is modified we need the expression for current.

The current operator in second-q. form

$$\hat{j}(r) = \frac{e}{m}\left(\psi^+(\hat{p} - \frac{e}{c}A)\psi - (\hat{p} + \frac{e}{c}A)\psi^+\psi\right)$$

Since we made the gauge transform $\psi \to e^{i\varphi/2}\psi$ we get

$$\hat{j}(r) = \frac{e}{m}\left(\psi^+ \hat{p}\psi - \hat{p}\psi^+\psi\right) + 2eV_s\psi^+\psi \quad \longrightarrow$$

The average is

$$\vec{j}(r) = \frac{2e}{m}(\hat{p}_1 + m\vec{V}_s)\, G(r_1, r_2, \tau = -0)\Big|_{r_1 = r_2}$$

$G(\tau=-0) = T\sum G(\omega_n)$ careful about $\tau \to 0$ limit

In momentum representation

$$\vec{j} = \frac{2e}{m} T \sum_{p,\omega_n} (\vec{p} + m\vec{V}_s)\, G(p, \omega_n)$$

Substitute here $G = \dfrac{\xi_p + i\tilde{\omega}_n}{\xi_p^2 + \hat{\omega}_n^2 + |\Delta|^2} = -\dfrac{1}{2}\left(\dfrac{1 + \xi/\mathcal{E}}{i\hat{\omega}_n - \mathcal{E}} + \dfrac{1 - \xi/\mathcal{E}}{i\hat{\omega}_n + \mathcal{E}}\right)$

To calculate the sum over Matsubara frequencies we use

$$T\sum_n \frac{e^{i+0\omega_n}}{i\omega_n - \alpha} = f_0(\alpha/T) \quad \text{where } f_0 = (e^x + 1)^{-1}$$

Then

$$j = \frac{e}{m}\int(p + m\vec{V}_s)\left[\left(1 + \frac{\xi}{\mathcal{E}}\right) f_0\left(\frac{\mathcal{E} + pV_s}{T}\right) + \left(1 - \frac{\xi}{\mathcal{E}}\right) \underbrace{f_0\left(\frac{-\mathcal{E} + pV_s}{T}\right)}_{1 - f_0\left(\frac{\mathcal{E} - pV_s}{T}\right)}\right] \frac{d^3p}{(2\pi)^3}$$

For simplicity send $T \to 0$ then $f_0 \to 0$

$$j = eNV_s, \quad \text{where } N = \int\frac{d^3k}{(2\pi)^3} = \text{density}.$$

+ Maxwell $\operatorname{rot} B = 4\pi j$:

If $B \parallel \cancel{x}$, $A \parallel y$ then $\dfrac{d^2}{dz^2}A = -4\pi j = -\dfrac{4\pi e N}{m}(\nabla\varphi - eA)$

$\lambda_L^2 = \left(\dfrac{m}{4\pi e^2 N}\right)$ - London length.

The average is

$$j(r) = -\frac{e}{m}\left[\hat{p}_1 - \hat{p}_2 + 2mV_s\right] G(r_1, r_2)\Big|_{r_1=r_2}$$

In the momentum representation

$$G(r_1,r_2) = \int G(p_1,p_2)\, e^{ip_1 r_1 - ip_2 r_2} \frac{d^3p_1\, d^3p_2}{(2\pi)^6}$$

$$j(k) = -\frac{2e}{m}\int (p + mV_s)\, G(p+k/2, p-k/2) \frac{d^3p}{(2\pi)^3}$$

In the homogeneous case

$$G(p+k/2, p-k/2) = G(p)\,\delta(k)$$

We assume that the scale of V_s is much longer than the coherence length - the scale of $G(p)$. Then $j(k)\,\delta(k) \to$ space-independent

Hence
$$j = -\frac{2e}{m}\int (p + mV_s)\, G(p) \frac{d^3p}{(2\pi)^3} = -\frac{2e}{m} T \sum_n \int (p+mV_s)\, G(p,\omega_n) \frac{d^3p}{(2\pi)^3}$$

Substituting

$$G = \frac{\xi_p + i\hat{\omega}_n}{\xi_p^2 + \hat{\omega}_n^2 + |\Delta|^2}, \quad \hat{\omega}_n = \omega_n + i p V_s$$

we will often use the relation

$$T\sum_n \frac{e^{+i0\omega_n}}{i\omega_n - \alpha} = f_0(\alpha/T), \quad \text{where} \quad f_0 = \frac{1}{e^x+1} - \text{Fermi distr.}$$

$$= \frac{1}{2}\left(1 - \operatorname{th}\frac{x}{2}\right)$$

If we choos $T\sum_n \frac{e^{-i0\omega_n}}{i\omega_n - \alpha} = f_0(\alpha/T) - 1$

a) The second term can be calculated as

$$G = T\sum_n \frac{\xi_p + i\omega_n}{\xi_p^2 + \omega_n^2 + |\Delta|^2} = -\frac{T}{2}\sum_n \left(\frac{1+\xi/\varepsilon}{i\omega_n - \varepsilon} + \frac{1-\xi/\varepsilon}{i\omega_n + \varepsilon}\right) = -\left(1+\xi/\varepsilon\right)\frac{f_0(\varepsilon/T)}{2} - \left(1-\xi/\varepsilon\right)\frac{f(-\varepsilon/T)}{2}$$

$$= -\frac{1}{2}\left(1 - \xi/\varepsilon\right) - \frac{\xi}{\varepsilon} f_0(\varepsilon/T)$$

Density of electrons $\quad n = -\underbrace{2}_{\text{spin}} G(\tau = -0)$

$$j_x = e n_0 V_s$$

$$\underbrace{\int d\xi\, \xi\, \frac{f(\varepsilon/T)}{\varepsilon} = 0}_{\text{odd functio}}$$

b) The first term

$$T\sum_n \int \bar{p}\, G(p,\omega_n) \frac{d^3p}{(2\pi)^3} = T\sum_n \int \frac{\bar{p}(\xi_p + i\hat{\omega}_n)}{\xi_p^2 + \hat{\omega}_n^2 + |\Delta|^2} \frac{d^3p}{(2\pi)^3}$$

We would like to substitute $\frac{d^3p}{(2\pi)^3} \to \frac{\sin\theta\, d\theta\, p^2\, dp}{(2\pi)^2} =$

$$= d\cos\theta\, \frac{\nu_0\, d\xi_p}{2}$$

where $\nu_0 = \frac{m p_F}{2\pi^2}$

We can take the integral by ξ_p taking into account the pole which is close to Fermi surface (4)

$$\int \frac{\xi_p + i\tilde{\omega}_n}{\xi_p^2 + \tilde{\omega}_n^2 + |\Delta|^2} \frac{d^3p}{(2\pi)^3} = \frac{\vartheta_0}{2} \int_{F.S.} \frac{\bar{p}(\xi_p + i\hat{\omega}_n)}{\xi_p^2 + \hat{\omega}_n^2 + |\Delta|^2} d\xi_p \, d\cos\theta \; +$$

$$+ \int \frac{\bar{P}}{\xi_p - i\tilde{\omega}_n} \frac{d^3p}{(2\pi)^3}$$

Off-shell contribution

Main contribution from the Fermi sea, when p far from Fermi surface.

$$\int_0^\infty p^3 dp \int_0^\pi \sin\theta \, d\theta \frac{\cos\theta}{\xi_p - i\omega_n + pV_s\cos\theta} = -\int_0^\infty p^3 dp \int_0^\pi \frac{pV_s \cos^2\theta \sin\theta \, d\theta}{(\xi_p - i\omega_n)^2}$$

$$= -\frac{2}{3} V_s \int_0^\infty \frac{p^4 dp}{(\xi_p - i\omega_n)^2}$$

$$\frac{T}{(2\pi)^2} \sum -\frac{2}{3} V_s \int_0^\infty \frac{p^4 dp}{(\xi_p - i\omega_n)^2} = -\frac{2}{3} \frac{V_s}{(2\pi)^2} \int_0 p^4 \frac{d}{d\xi_p} f_0\left(\xi_p/T\right) dp = -\frac{2m}{3} V_s \frac{p_F^3}{(2\pi)^2} = -\frac{nmV_s}{2}$$

Since $n = \frac{p_F^3}{3\pi^2}$

Hence finally

$$T \sum_n \int \bar{p}\, G(p, \omega_n) \frac{d^3p}{(2\pi)^3} = i\pi\vartheta_0 \sum_n \int_0^\pi \frac{\vec{p}_F \hat{\omega}_n T}{(\hat{\omega}_n^2 + |\Delta|^2)^{1/2}} d\cos\theta - en V_s$$

The current

$$j = -\frac{2ie}{m} \pi\vartheta_0 \sum_n \int_0^\pi \frac{\vec{p}_F \tilde{\omega}_n T}{(\tilde{\omega}_n^2 + |\Delta|^2)^{1/2}} d\cos\theta \qquad \hat{\omega}_n = \omega_n + i\vec{p}_F\vec{V}_s$$

Expand up to the first order in V_s

$$\frac{\bar{p}_F \hat{\omega}_n}{(\hat{\omega}_n^2 + |\Delta|^2)^{1/2}} = \frac{\vec{p}_F \omega_n}{(\omega_n^2 + |\Delta|^2)^{1/2}} + \frac{\vec{p}_F\, i(\vec{p}_F\vec{V}_s)}{(\omega_n^2 + |\Delta|^2)^{1/2}} - \frac{i\vec{p}_F \omega_n^2 (\vec{p}_F\vec{V}_s)}{(\omega_n^2 + |\Delta|^2)^{3/2}} \Rightarrow$$

$$\Rightarrow \frac{i\, p_F (p_F V_s) |\Delta|^2}{(\omega_n^2 + |\Delta|^2)^{3/2}}$$

$$j = +\frac{2e}{m} \pi\vartheta_0 V_s p_F^2 \, 2\int_0^1 x^2 dx \sum_n \frac{T|\Delta|^2}{(\omega_n^2 + |\Delta|^2)^{3/2}} = +\frac{4}{3} \frac{e}{m} \pi \vartheta_0 p_F^2 \sum_n \frac{T|\Delta|^2}{(\omega_n^2 + |\Delta|^2)^{3/2}} V_s$$

Recall that $\nu_0 = \frac{m p_F}{2\pi \hbar^2}$ then $\frac{2\nu_0 p_F^2}{3m} = n_0$

So that

$$j = 2e n_0 \sum_n \frac{T|\Delta|^2\pi}{(\omega_n^2+|\Delta|^2)^{3/2}} V_s = +\frac{2e\cdot n_0}{2m}\left(\nabla\varphi - \frac{2e}{\hbar c}A\right) \sum \frac{T|\Delta|^2\pi}{(\omega_n^2+|\Delta|^2)^{3/2}}$$

$$= \frac{2e^2 n_0}{\hbar c m}\left(\frac{\hbar c}{2e}\nabla\varphi - A\right) \sum \frac{T|\Delta|^2\pi}{(\omega_n^2+|\Delta|^2)^{3/2}} = \frac{c}{4\pi}\lambda_L^{-2}\left(\frac{\hbar c}{2e}\nabla\varphi - A\right)$$

where $\lambda_L^{-2} = \frac{4\pi n_s e^2}{mc^2}$

$n_s = n_0 |\Delta|^2 \cdot \sum_n \frac{\pi T}{(\omega_n^2+|\Delta|^2)^{3/2}}$ — density of superconducting component

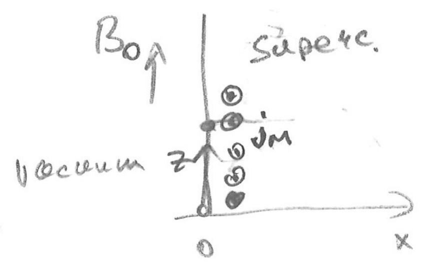
so that $j = e n_s V_s$ — definition of n_s

London model $\nabla\varphi = 0$ then $j = -\frac{c}{4\pi}\lambda_L^{-2} A$

Maxwell eq. $\text{rot}\, B = -\lambda_L^{-2} A$

$\text{rot rot}\, A = -\lambda_L^{-2} A$

Let's consider 1-d problem



$\vec{A} = A_y(x)\,\vec{y}$ $\quad$ $\text{rot rot}\, A = -\vec{y}\,\frac{d^2}{dx^2} A_y$

$B_z = \frac{\partial A_y}{\partial x}$

$$\frac{d^2}{dx^2} A_y - \lambda_L^{-2} A_y = 0$$

$$\begin{vmatrix} \bar{x} & \bar{y} & \bar{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & A_y & 0 \end{vmatrix}$$

$$A_y = A_y^0 e^{-x/\lambda_L}$$

$$B_z = B_0 e^{-x/\lambda_L}$$

gauge field
finite mass

Meissner effect

Physically that means that when we apply magnetic field it induces current along the surface. The field created by this current exactly compensates external field

Density of superconducting conpenent

$$n_s = n_0 |\Delta|^2 \sum_n \frac{\pi T}{(\omega_n^2 + |\Delta|^2)^{3/2}}$$

Let's consider two liminating cases

a) $T \to T_c$ $n_s = n_0 |\Delta|^2 \sum \frac{\pi T}{\omega_n^3} = n_0 |\Delta|^2 \sum \frac{1}{(2n+1)^3 (\pi T)^2} =$

$|\Delta| \ll \omega_n$

$$= n_0 \frac{|\Delta|^2}{T_c^2} \frac{7\zeta(3)}{8\pi^2} \sim (1 - T/T_c) \qquad d_n \to \infty$$

b) $T \to 0$ $\frac{n_s}{n_0} = |\Delta|^2 \int_{-\infty}^{\infty} \frac{\pi T \, dn}{((2n+1)^2 (\pi T)^2 + |\Delta|^2)^{3/2}} = \frac{1}{2}\int_0^{\infty} \frac{dx}{(x^2+1)^{3/2}} = 1$

$x = \operatorname{sh} t$ $\int \frac{dt}{\operatorname{ch}^2 t} = \operatorname{th} t$

$dx = \operatorname{ch} t \, dt$

$(x^2+1)^{3/2} = \operatorname{ch}^3 t$

Flux quantization.

London approximation $\nabla \varphi = 0$ is not always correct

Phase gradient can compensate the vector potential

Consider multiply connected domain.

Due to the Meissner effect there is no current in the bulk $j_s = 0$

$$\frac{\hbar c}{2e} \nabla \varphi - A = 0$$

$$\oint \nabla \varphi = 2\pi n \quad \Rightarrow \quad \oint A\, dl = \Phi = \frac{\hbar c}{2e} \cdot 2\pi n = \frac{\pi \hbar c}{e} n$$

$$\Phi_0 = \frac{\pi \hbar c}{e}$$ = magnetic flux quantum

Magnetic field is confined within non-superconducting domain. Such domains can be created either artificially just drilling holes in superconductor. Also they can appear naturally when the non-superconducting region is just the normal metal. Such normal tubes with localized magnetic flux are <u>Abrikosov vortices</u>