

Introduction to Bose systems

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i) Bosonic operators $[\alpha_\alpha, \alpha_\beta^+] = \delta_{\alpha\beta}$ $[\alpha_\alpha, \alpha_\beta] = [\alpha_\alpha^+, \alpha_\beta^+] = 0$

It changes completely the system properties and construction of perturbation theory.

Let's start with a review of non-interacting system

Ideal Bose gas

$$\hat{H} = \sum \epsilon_k \hat{a}_k^\dagger \hat{a}_k$$

Let's find distribution function in a grand canonical ensemble

$$\text{Density matrix } \hat{\rho} = \exp(\beta(\mathcal{H} + \mu N - H))$$

The thermodynamic potential is defined from normalization

$$\text{Tr } \hat{\rho} = 1 \Rightarrow \mathcal{Z} = -T \ln \text{Tr} [\exp(\beta(\mu N - H))]$$

$$\text{Tr } A = \sum \langle n | A | n \rangle$$

Consider one of the states in Fock space $|n\rangle = |\text{macrostate}\rangle$

$$\langle n | \exp(\beta(\mu N - H)) | n \rangle = \exp\left(\beta \sum_k (\mu - \epsilon_k) n_k\right)$$

Each state can be occupied by arbitrary number of particles

$$\mathcal{Z} = -T \ln \left[\sum_{nk} \exp\left(\beta \left(\sum_k (\mu - \epsilon_k) n_k \right)\right) \right]$$

$$\text{Consider one term } k=1 \quad \sum_n \exp(\beta(\mu - \epsilon_1)n_1) = \frac{1}{1 - e^{\beta(\mu - \epsilon_1)}}$$

This sum converges only

$$\text{if } \mu < \epsilon_1 \Rightarrow \mu \leftarrow 0$$

$$\text{Then } \mathcal{Z} = \sum_k \mathcal{Z}_k \text{ where } \mathcal{Z}_k = -\ln(1 - e^{\beta(\mu - \epsilon_k)})$$

$$\text{Distribution function } n_B(p) = -\frac{\partial \mathcal{Z}_k}{\partial \mu} = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

$$\text{Total particle number } N = \sum_p n_B(p)$$

As well known from the statistical mechanics (2) course the first term with $k=0$ is singular. Because at $\mu \rightarrow 0$ $n_B(0) \rightarrow \infty$. At $T \rightarrow T_c$ all particles will sit at $k=0$ state.

This makes qualitative difference with Fermi systems when we apply the perturbation theory. Indeed the ground state at $T=0$ has zero energy $E=0$. Then arbitrary small interaction will be large as compared to the ground state energy so that the perturbation theory doesn't work. Smarter approach is needed and it was suggested by Bogoliubov.

Weakly interacting Bose gas

$$H_{\text{int}} = \frac{1}{2} \int g(r-r') \psi^+(r) \psi^+(r') \psi(r') \psi(r) d^3r d^3r'$$

which can be written in momentum representation

$$H_{\text{int}} = \frac{g}{2V} \sum_{p p' q} a_p^+ a_{p'}^+ a_{p+q} a_{p-q}$$

According to Bogoliubov idea the average number of particles in the ground state N_0 is large and is close to the total particle number N

$$\langle a_0 | N_0 \rangle = \sqrt{N_0} | N_0 - 1 \rangle$$

$$a_0^+ | N_0 \rangle = \sqrt{N_0 + 1} | N_0 + 1 \rangle$$

And substitute them by numbers. $a_0 = a_0^+ = \sqrt{N_0}$
 because in Hamiltonian only the terms which contain a_p , a_p^+ have $\frac{g}{2V} \sum_{p p' q} a_p^+ a_{p'}^+ a_{p+q} a_{p-q} = \frac{g N_0}{V} \sum_p a_p^+ a_p + \frac{g N_0}{2V} \sum_p (a_p^+ a_{p+} + a_p a_{-p}) + \frac{g N_0^2}{2V}$

Hence the Hamiltonian has a familiar form

$$H = \sum_p \frac{p^2}{2m} a_p^+ a_p + \frac{g N_0}{V} \sum_{p \neq 0} a_p^+ a_p + \frac{g N_0}{2V} \sum_{p > 0} (a_p^+ a_{-p} + a_p a_{-p}) + \frac{g N_0^2}{2V}$$

looks almost the same as BCS mean field model
 can be diagonalized in the similar way of Bogoliubov transformation

We see that the Hamiltonian (3)

$$H = \sum_{p \neq 0} H_p$$

$$H_p = \xi_p (a_p^+ a_p + a_{-p}^+ a_p) + g n_0 (a_p^+ a_{-p}^+ + a_p a_{-p})$$

$$\text{where } \xi_p = \frac{p^2}{2m} + g n_0, \quad n_0 = N_0/V$$

Bogoliubov transform $\begin{cases} a_p = u_p d_p - v_p d_{-p} \\ a_{-p} = -v_p d_p + u_p d_{-p} \end{cases}$ assume that $u_p, v_p = \text{real}$, $u_p = u_{-p}$, $v_p = v_{-p}$

$$\begin{aligned} [a_p, a_p^+] &= (u_p d_p - v_p d_{-p})(u_p d_p^+ - v_p d_{-p}^+) - (u_p d_p^+ - v_p d_{-p}^*)(u_p d_p - v_p d_{-p}) = \\ &= u_p^2 d_p d_p^+ - u_p v_p d_p d_{-p} - v_p u_p d_{-p}^+ d_p^+ + v_p^2 d_{-p}^+ d_{-p} = \\ &= u_p^2 d_p^+ d_p + v_p u_p d_{-p} d_p + u_p v_p d_p^+ d_{-p}^+ - v_p^2 d_{-p} d_{-p}^+ = \\ &= u_p^2 (d_p d_p^+ - d_p^+ d_p) + v_p^2 (d_{-p}^+ d_{-p} - d_{-p} d_{-p}^+) + u_p v_p (d_{-p} d_p - d_{-p}^+ d_p) + \\ &\quad + u_p v_p (d_p^+ d_{-p}^+ - d_{-p}^+ d_p^+) = u_p^2 - v_p^2 = 1 \quad \text{Normalization} \end{aligned}$$

What happens with Hamiltonian?

$$H_p = \xi_p \left[(u_p d_p^+ - v_p d_{-p})(u_p d_p - v_p d_{-p}^+) + (u_p d_{-p}^+ - v_p d_p)(u_p d_{-p} - v_p d_p^+) \right] + g n_0 \left[(u_p d_p^+ - v_p d_{-p})(u_p d_{-p}^+ - v_p d_p) + (u_p d_p - v_p d_{-p}^+)(u_p d_{-p} - v_p d_p^+) \right]$$

$$\begin{aligned} \text{kinetic energy} &= \xi_p \left[u_p^2 \cancel{d_p^+ d_p} + v_p^2 \cancel{d_{-p} d_{-p}^+} - u_p v_p (\cancel{d_p^+ d_{-p}} + \cancel{d_{-p} d_p}) + \right. \\ &\quad \left. + u_p^2 \cancel{d_{-p}^+ d_{-p}} + v_p^2 \cancel{d_p d_p^+} - u_p v_p (\cancel{d_{-p}^+ d_p} + \cancel{d_p d_{-p}}) \right] = \\ &= \xi_p (u_p^2 + v_p^2) (\cancel{d_p^+ d_p} + \cancel{d_{-p}^+ d_{-p}}) + 2 \xi_p v_p^2 - 2 \xi_p u_p v_p (\cancel{d_p^+ d_p} + \cancel{d_p d_{-p}}) \end{aligned}$$

$$\begin{aligned} \text{Interaction energy} &= g n_0 \left[u_p^2 \cancel{d_p^+ d_{-p}^+} + v_p^2 \cancel{d_{-p}^+ d_p^+} - u_p v_p (\cancel{d_p^+ d_{-p}^+} + \cancel{d_{-p}^+ d_p^+}) + \right. \\ &\quad \left. + u_p^2 d_p d_{-p} + v_p^2 d_{-p} d_p - u_p v_p (d_p d_{-p}^+ + d_{-p}^+ d_p) \right] = \\ &= g n_0 (u_p^2 + v_p^2) (\cancel{d_p^+ d_{-p}^+} + \cancel{d_p d_{-p}}) - 2 g n_0 u_p v_p (d_p^+ d_p + d_{-p}^+ d_{-p}) - 2 g n_0 u_p v_p \end{aligned}$$

$$\begin{aligned} \text{Thus } H_p &= \left[\xi_p (u_p^2 + v_p^2) - 2 g n_0 u_p v_p \right] (\cancel{d_p^+ d_p} + \cancel{d_{-p}^+ d_{-p}}) + \\ &\quad + \left[g n_0 (u_p^2 + v_p^2) - 2 \xi_p u_p v_p \right] (d_p^+ d_{-p}^+ + d_p d_{-p}) + 2 \xi_p v_p^2 - 2 g n_0 u_p v_p \end{aligned}$$

The Hamiltonian becomes diagonal if

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$$g\hbar\omega(u_p^2 + v_p^2) - 2\varepsilon_p u_p v_p = 0$$

$$u_p^2 - v_p^2 = g$$

We obtain quadratic equation for u_p

$$g\hbar\omega(2u_p^2 + 1) - 2\varepsilon_p u_p(u_p^2 - 1)^{1/2} = 0$$

The solution is straightforward

$$u_p^2 = \frac{1}{2}(1 + \frac{\varepsilon_p}{\varepsilon_p}) \quad v_p^2 = \frac{1}{2}(\frac{\varepsilon_p}{\varepsilon_p} - 1)$$

where $\varepsilon_p = \sqrt{\varepsilon_p^2 - (g\hbar\omega)^2} = \sqrt{(\frac{p^2}{2m} + g\hbar\omega)^2 - (g\hbar\omega)^2} = p\sqrt{\frac{g\hbar\omega}{m} + \frac{p^2}{4m^2}}$

is the excitation energy since

$$H_p = \varepsilon_p \frac{d^2}{dp^2} + 2\varepsilon_p v_p^2 - 2g\hbar\omega u_p v_p$$

Properties of the excitation spectrum

1) At $p \rightarrow 0$ we get sound waves $E = p\sqrt{\frac{g\hbar\omega}{m}}$ $S = \sqrt{\frac{g\hbar\omega}{m}} = \text{sound velocity}$

2) At $p \gg g\hbar\omega$ the spectrum of free particles $E = \frac{p^2}{2m}$

3) Ground state energy: when $2\varepsilon_p > 0 \Rightarrow 0$

$$E_0 = \frac{gN_0^2}{2V} + \frac{1}{p} (2\varepsilon_p v_p^2 - 2g\hbar\omega u_p v_p) = \frac{gN_0^2}{2V} + \frac{1}{p} (\varepsilon_p - \varepsilon_p)$$

The second term is much smaller if $g \ll \frac{\hbar^2}{h^3 m}$ although calcul. is non-trivial

a) Chemical potential $\mu = \frac{\partial E_0}{\partial N} = g\hbar\omega$ - energy per one particle

b) pressure $P = -\frac{\partial E_0}{\partial V} = \frac{g\hbar\omega^2}{2}$

c) sound velocity $S = \sqrt{\frac{\partial P}{\partial \rho}} = \sqrt{\frac{g\hbar\omega}{m}}$ coincides with that obtained from spectrum

II) Superfluidity

Non-interacting Bose condensate is not superfluid. It is possible to excite particles paying arbitrarily small energy

Free spectrum at $g=0$ $E = \frac{p^2}{2m^2}$ - does not lead to superfluidity

For interacting gas

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$$\text{Cylinder} \rightarrow \vec{v}$$

$$\vec{r} = \frac{\vec{P}^2}{2m}$$

$B_p = B_{p-q} + \epsilon_q$ - conservation of energy
and momentum

$$B_p - B_{p-q} = \frac{\partial B}{\partial P} q = \vec{v} \cdot \vec{q}$$

$$\vec{v} \cdot \vec{q} = \sqrt{s^2 q^2 + \frac{q^4}{m^2}}$$

If $N < S$ no solution
the motion is without friction