

# Introduction to Bose systems

(1)

Bosonic operators  $[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$   $[a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0$

It changes completely the system properties and construction of perturbation theory.

Let's start with a refresher of non-interacting system

## Ideal Bose gas

$$\hat{H} = \sum \epsilon_k a_k^\dagger a_k$$

Let's find distribution function in a grand canonical ensemble

Density matrix  $\hat{\rho} = \exp(\beta(\Omega + \mu N - H))$

Thermodynamic potential is defined from normalization

$$\text{Tr } \hat{\rho} = 1 \Rightarrow \Omega = -T \ln \text{Tr} [\exp(\beta(\mu N - H))]$$

$$\text{Tr } A = \sum \langle n | A | n \rangle$$

Consider one of the states in Fock space  $|n\rangle = |n_1, n_2, \dots\rangle$

$$\langle n | \exp(\beta(\mu N - H)) | n \rangle = \exp(\beta \sum_k (\mu - \epsilon_k) n_k)$$

Each state can be occupied by arbitrary number of particles

$$\Omega = -T \ln \left[ \sum_{\{n_k\}} \exp(\beta \sum_k (\mu - \epsilon_k) n_k) \right]$$

Consider one term  $k=1$   $\sum_{n_1} \exp(\beta(\mu - \epsilon_1) n_1) = \frac{1}{1 - e^{\beta(\mu - \epsilon_1)}}$

This sum converges only

if  $\mu < \epsilon_1 \Rightarrow \mu < 0$

Then  $\Omega = \sum_k \Omega_k$  where  $\Omega_k = -T \ln(1 - e^{\beta(\mu - \epsilon_k)})$

Distribution function Bose  $n_B(p) = -\frac{\partial \Omega_k}{\partial \mu} = \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1}$

Total particle number  $N = \sum_p n_B(p)$

As well known from the statistical mechanics (2) source the first term with  $k=0$  is singular. Because at  $\mu \rightarrow 0$   $n_B(0) \rightarrow \infty$ , At  $T < T_0$  all particles will sit at  $k=0$  state.

This makes qualitative difference with Fermi systems when we apply the perturbation theory. Indeed the ground state at  $T=0$  has zero energy  $E=0$ . Then an arbitrary small interaction will be large as compared to the ground state energy so that the perturbation theory doesn't work. Smarter approach is needed and it was suggested by Bogolubov.

### Weakly interacting Bose gas

$$H_{int} = \frac{1}{2} \int g(r-r') \psi^\dagger(r) \psi^\dagger(r') \psi(r') \psi(r) d^3r d^3r'$$

which can be written in momentum representation

$$H_{int} = \frac{g}{2V} \sum_{pp'q} a_p^\dagger a_{p'}^\dagger a_{p'+q} a_{p-q}$$

According to Bogolubov idea the average number of particles in the ground state  $N_0$  is large and is close to the total particle number  $N$ .

One can separate operator  $a_0 |N_0\rangle = \sqrt{N_0} |N_0-1\rangle$   
 $a_0^\dagger |N_0\rangle = \sqrt{N_0+1} |N_0+1\rangle$

And substitute them by numbers.  $a_0 = a_0^\dagger = \sqrt{N_0}$   
 hence in Hamiltonian only the terms which contain  $a_{p0}, a_{p0}^\dagger$   
 $\frac{g}{2V} \sum_{pp'q} a_p^\dagger a_{p'}^\dagger a_{p'+q} a_{p-q} = \frac{gN_0}{V} \sum_p a_p^\dagger a_p + \frac{gN_0}{2V} \sum_p (a_p^\dagger a_{-p} + a_p a_{-p}) + \frac{gN_0^2}{2V}$

Hence the Hamiltonian has a familiar form

$$H = \sum_p \frac{p^2}{2m} a_p^\dagger a_p + \frac{gN_0}{V} \sum_{p \neq 0} a_p^\dagger a_p + \frac{gN_0}{2V} \sum_{p \neq 0} (a_p^\dagger a_{-p} + a_p a_{-p}) + \frac{gN_0^2}{2V}$$

looks almost the same as BCS mean field model can be diagonalized in the similar way of Bogolubov transformation

We see that the Hamiltonian

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$$H = \sum_{\mathbf{p} \neq 0} H_{\mathbf{p}}$$

$$H_{\mathbf{p}} = \xi_{\mathbf{p}} (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger} a_{-\mathbf{p}}) + g n_0 (a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} + a_{\mathbf{p}} a_{-\mathbf{p}})$$

where  $\xi_{\mathbf{p}} = \frac{p^2}{2m} + g n_0$ ,  $n_0 = N_0/V$

Bogolubov transform  $\begin{cases} a_{\mathbf{p}} = U_{\mathbf{p}} d_{\mathbf{p}} - V_{\mathbf{p}} d_{-\mathbf{p}}^{\dagger} \\ a_{-\mathbf{p}}^{\dagger} = -V_{\mathbf{p}} d_{\mathbf{p}} + U_{\mathbf{p}} d_{-\mathbf{p}}^{\dagger} \end{cases}$

assume that  $U_{\mathbf{p}}, V_{\mathbf{p}} = \text{real}$ ,  $U_{\mathbf{p}} = U_{-\mathbf{p}}$ ,  $V_{\mathbf{p}} = V_{-\mathbf{p}}$

$$\begin{aligned} [a_{\mathbf{p}}, a_{\mathbf{p}}^{\dagger}] &= (U_{\mathbf{p}} d_{\mathbf{p}} - V_{\mathbf{p}} d_{-\mathbf{p}}^{\dagger}) (U_{\mathbf{p}} d_{\mathbf{p}}^{\dagger} - V_{\mathbf{p}} d_{-\mathbf{p}}) - (U_{\mathbf{p}} d_{\mathbf{p}}^{\dagger} - V_{\mathbf{p}} d_{-\mathbf{p}}) (U_{\mathbf{p}} d_{\mathbf{p}} - V_{\mathbf{p}} d_{-\mathbf{p}}^{\dagger}) \\ &= U_{\mathbf{p}}^2 d_{\mathbf{p}} d_{\mathbf{p}}^{\dagger} - U_{\mathbf{p}} V_{\mathbf{p}} d_{\mathbf{p}} d_{-\mathbf{p}} - V_{\mathbf{p}} U_{\mathbf{p}} d_{-\mathbf{p}}^{\dagger} d_{\mathbf{p}}^{\dagger} + V_{\mathbf{p}}^2 d_{-\mathbf{p}}^{\dagger} d_{-\mathbf{p}} - \\ &\quad - U_{\mathbf{p}}^2 d_{\mathbf{p}}^{\dagger} d_{\mathbf{p}} + V_{\mathbf{p}} U_{\mathbf{p}} d_{\mathbf{p}} d_{\mathbf{p}} + U_{\mathbf{p}} V_{\mathbf{p}} d_{\mathbf{p}}^{\dagger} d_{-\mathbf{p}}^{\dagger} - V_{\mathbf{p}}^2 d_{-\mathbf{p}} d_{-\mathbf{p}}^{\dagger} = \\ &= U_{\mathbf{p}}^2 (d_{\mathbf{p}} d_{\mathbf{p}}^{\dagger} - d_{\mathbf{p}}^{\dagger} d_{\mathbf{p}}) + V_{\mathbf{p}}^2 (d_{-\mathbf{p}}^{\dagger} d_{-\mathbf{p}} - d_{-\mathbf{p}} d_{-\mathbf{p}}^{\dagger}) + U_{\mathbf{p}} V_{\mathbf{p}} (d_{-\mathbf{p}} d_{\mathbf{p}} - d_{-\mathbf{p}} d_{\mathbf{p}}) + \\ &\quad + U_{\mathbf{p}} V_{\mathbf{p}} (d_{\mathbf{p}}^{\dagger} d_{-\mathbf{p}}^{\dagger} - d_{-\mathbf{p}}^{\dagger} d_{\mathbf{p}}^{\dagger}) = U_{\mathbf{p}}^2 - V_{\mathbf{p}}^2 \stackrel{!}{=} 1 \quad \text{Normalization} \end{aligned}$$

What happens with Hamiltonian?

$$H_{\mathbf{p}} = \xi_{\mathbf{p}} \left[ (U_{\mathbf{p}} d_{\mathbf{p}}^{\dagger} - V_{\mathbf{p}} d_{-\mathbf{p}}) (U_{\mathbf{p}} d_{\mathbf{p}} - V_{\mathbf{p}} d_{-\mathbf{p}}^{\dagger}) + (U_{\mathbf{p}} d_{-\mathbf{p}}^{\dagger} - V_{\mathbf{p}} d_{\mathbf{p}}) (U_{\mathbf{p}} d_{-\mathbf{p}} - V_{\mathbf{p}} d_{\mathbf{p}}^{\dagger}) \right] + g n_0 \left[ (U_{\mathbf{p}} d_{\mathbf{p}}^{\dagger} - V_{\mathbf{p}} d_{-\mathbf{p}}) (U_{\mathbf{p}} d_{-\mathbf{p}}^{\dagger} - V_{\mathbf{p}} d_{\mathbf{p}}) + (U_{\mathbf{p}} d_{-\mathbf{p}} - V_{\mathbf{p}} d_{\mathbf{p}}^{\dagger}) (U_{\mathbf{p}} d_{\mathbf{p}} - V_{\mathbf{p}} d_{-\mathbf{p}}^{\dagger}) \right]$$

kinetic energy  $= \xi_{\mathbf{p}} \left[ U_{\mathbf{p}}^2 d_{\mathbf{p}}^{\dagger} d_{\mathbf{p}} + V_{\mathbf{p}}^2 d_{-\mathbf{p}} d_{-\mathbf{p}}^{\dagger} - U_{\mathbf{p}} V_{\mathbf{p}} (d_{\mathbf{p}}^{\dagger} d_{-\mathbf{p}}^{\dagger} + d_{-\mathbf{p}} d_{\mathbf{p}}) + U_{\mathbf{p}}^2 d_{-\mathbf{p}}^{\dagger} d_{-\mathbf{p}} + V_{\mathbf{p}}^2 d_{\mathbf{p}} d_{\mathbf{p}}^{\dagger} - U_{\mathbf{p}} V_{\mathbf{p}} (d_{-\mathbf{p}}^{\dagger} d_{\mathbf{p}}^{\dagger} + d_{\mathbf{p}} d_{-\mathbf{p}}) \right] =$

$$= \xi_{\mathbf{p}} (U_{\mathbf{p}}^2 + V_{\mathbf{p}}^2) (d_{\mathbf{p}}^{\dagger} d_{\mathbf{p}} + d_{-\mathbf{p}}^{\dagger} d_{-\mathbf{p}}) + 2 \xi_{\mathbf{p}} V_{\mathbf{p}}^2 - 2 \xi_{\mathbf{p}} U_{\mathbf{p}} V_{\mathbf{p}} (d_{\mathbf{p}}^{\dagger} d_{\mathbf{p}}^{\dagger} + d_{\mathbf{p}} d_{-\mathbf{p}})$$

Interaction energy  $= g n_0 \left[ U_{\mathbf{p}}^2 d_{\mathbf{p}}^{\dagger} d_{-\mathbf{p}}^{\dagger} + V_{\mathbf{p}}^2 d_{-\mathbf{p}} d_{\mathbf{p}} - U_{\mathbf{p}} V_{\mathbf{p}} (d_{\mathbf{p}}^{\dagger} d_{\mathbf{p}} + d_{-\mathbf{p}} d_{-\mathbf{p}}^{\dagger}) + U_{\mathbf{p}}^2 d_{\mathbf{p}} d_{-\mathbf{p}} + V_{\mathbf{p}}^2 d_{-\mathbf{p}}^{\dagger} d_{\mathbf{p}}^{\dagger} - U_{\mathbf{p}} V_{\mathbf{p}} (d_{\mathbf{p}} d_{\mathbf{p}}^{\dagger} + d_{-\mathbf{p}}^{\dagger} d_{-\mathbf{p}}) \right] =$

$$= g n_0 (U_{\mathbf{p}}^2 + V_{\mathbf{p}}^2) (d_{\mathbf{p}}^{\dagger} d_{-\mathbf{p}}^{\dagger} + d_{\mathbf{p}} d_{-\mathbf{p}}) - 2 g n_0 U_{\mathbf{p}} V_{\mathbf{p}} (d_{\mathbf{p}}^{\dagger} d_{\mathbf{p}} + d_{-\mathbf{p}}^{\dagger} d_{-\mathbf{p}}) - 2 g n_0 U_{\mathbf{p}} V_{\mathbf{p}}$$

Thus  $H_{\mathbf{p}} = \left[ \xi_{\mathbf{p}} (U_{\mathbf{p}}^2 + V_{\mathbf{p}}^2) + 2 g n_0 U_{\mathbf{p}} V_{\mathbf{p}} \right] (d_{\mathbf{p}}^{\dagger} d_{\mathbf{p}} + d_{-\mathbf{p}}^{\dagger} d_{-\mathbf{p}}) +$

$$+ \left[ g n_0 (U_{\mathbf{p}}^2 + V_{\mathbf{p}}^2) - 2 \xi_{\mathbf{p}} U_{\mathbf{p}} V_{\mathbf{p}} \right] (d_{\mathbf{p}}^{\dagger} d_{\mathbf{p}}^{\dagger} + d_{\mathbf{p}} d_{-\mathbf{p}}) + 2 \xi_{\mathbf{p}} V_{\mathbf{p}}^2 - 2 g n_0 U_{\mathbf{p}} V_{\mathbf{p}}$$

The Hamiltonian becomes diagonal if

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$$g n_0 (u_p^2 + v_p^2) - 2 \xi_p u_p v_p = 0$$

$$u_p^2 - v_p^2 = 1$$

We obtain quadratic equation for  $u_p$

$$g n_0 (2u_p^2 + 1) - 2 \xi_p u_p (u_p^2 - 1)^{1/2} = 0$$

The solution is straight forward

$$u_p^2 = \frac{1}{2} \left( 1 + \frac{\xi_p}{\epsilon_p} \right) \quad v_p^2 = \frac{1}{2} \left( \frac{\xi_p}{\epsilon_p} - 1 \right)$$

where  $\epsilon_p = \sqrt{\xi_p^2 - (g n_0)^2} = \sqrt{\left( \frac{p^2}{2m} + g n_0 \right)^2 - (g n_0)^2} = p \sqrt{\frac{g n_0}{m} + \frac{p^2}{4m^2}}$

is the excitation energy since

$$H_p = \epsilon_p a_p^\dagger a_p + 2 \xi_p v_p^2 - 2 g n_0 u_p v_p$$

Properties of the excitation spectrum

1) At  $p \rightarrow 0$  we get sound waves  $\epsilon = p \sqrt{\frac{g n_0}{m}}$   $s = \sqrt{\frac{g n_0}{m}} = \text{sound velocity}$

2) At  $p \gg g n_0$  the spectrum of free particles  $\epsilon = \frac{p^2}{2m}$

3) Ground state energy: when  $\langle p|0 \rangle = 0$

$$E_0 = \frac{g N_0^2}{2V} + \sum_p (2 \xi_p v_p^2 - 2 g n_0 u_p v_p) = \frac{g N_0^2}{2V} + \sum_p (\epsilon_p - \xi_p)$$

The second term is much smaller if  $g \ll \frac{\hbar^2}{m^{1/3}}$  although calcul. is non-trivial

a) Chemical potential  $\mu = \frac{\partial E_0}{\partial N} = g n_0$  - energy per one particle

b) pressure  $p = -\frac{\partial E_0}{\partial V} = \frac{g n_0^2}{2}$

c) sound velocity  $s = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{\frac{g n_0}{m}}$  coincides with that obtained from spectrum

4) Superfluidity

Non-interacting Bose condensate is not superfluid. It is possible to excite particles paying arbitrary small energy

Free spectrum at  $g=0$   $\epsilon = \frac{p^2}{2m^2}$  - does not lead to superfluidity

For interacting gas



$$\vec{E} = \frac{p^2}{2m}$$

$$\vec{E}_p = \vec{E}_{p-q} + \epsilon_q \quad - \text{conservation of energy and momentum}$$

$$\vec{E}_p - \vec{E}_{p-q} = \frac{\partial B}{\partial p} q = \vec{v}_q$$

$$\vec{v}_q = \sqrt{s^2 q^2 + \frac{q^4}{4m^2}}$$

If  $V < S$  no solution  
the motion is without friction