1. Introduction. The concept of quasiparticles.

Strictly speaking the only solvable problems in physics are just one-body problems. This is because the two-body problem can be reduced to the one-body case. The three-body problem is already unsolvable. – only numerically

However the things we deal with in condensed matter physics usually consist of a huge number of particles. For example a cubic centimeter of gas or solid can contain of the order $10^{19}-10^{23}$ particles. Examples →

air (any gas at normal conditions) $N_a/22 \cdot 10^3 \approx 10^{19}$

Moreover many systems that we meet in everyday life, e.g. metals have enough density for the particles to be very close to each other → much closer than de Broglei wave lenght. Therefore a theory is needed to describe quantum behaviour of the systems containing huge amount of particles.

In this case it is technically impossible to determine the behavior of each particle individually. On the other hand macroscopic properties are determined by collective behavior which can be characterized by only a few variables.

The basic idea of such approach is that instead of following a large number of strongly interacting real particles we should try to get away with considering relatively small number of weakly interacting quasiparticles or elementary excitations.

An elementary excitation is what its name implies: something that appears in the system after it has suffered an external perturbation. In a crystal lattice such excitations are phonons, sound quanta.

The phonons form rather dilute Bose gas therefore they are much easier to deal with than actual particles: atoms in lattice because they can be considered as non-interacting particles.
The phonons are called quasiparticles not only because they do not exist without the crystal lattice. Most importantly they have finite life time in contrast to the usual particles.
A key point here is that quasiparticles must be stable enough. If they decay faster than they can be created the description loses sense.
The life time of a quasiparticle is determined by its interactions with other quasiparticles. Interactions between quasiparticles can be completely different from that of actual particles. To convince you in this I would like to consider a system of interacting electrons in a metal lattice. The lattice consists of positive ions and we will consider it as uniformely distributed positive charge neutralizing the total charge of free electrons.
Then we have a huge number of electrons which interact with each other throug Coulomb force $\sim 1/r^2$ — it has infinite radius. Therefore to calculate electrostatic field created by given electron we have to take into account responce of all other electrons.
Since there are almost infinite number of them - nothing depends on the behaviour of individual electrons. In this case we can describe their action by

average electron density $n(\vec{r})$

Such approach is called the

Mean field approximation (MPA)

Screening of Coulomb Potential in Metals.

Suppose we place external charge Q in the system. It will create potential $\varphi(r)$ which will change the initial uniform distribution of electronic density

$$n = \frac{p_F^3}{3\pi^2\hbar^3}, \text{ where } p_F = \sqrt{2m\mu} \text{ is the Fermi moment.}$$

When the electron density becomes coordinate-dependent so does the Fermi momentum $p_F \to p_F(n(\vec{r}))$

. In equilibrium the electrochemical potential of the electrons must be constant

$$\mu = \frac{p_F^2(n(r))}{2m} + e\Phi(r) = \text{const}$$

$$n(r) = \frac{\left[2m(\mu - e\Phi(r))\right]^{3/2}}{3\pi^2\hbar^3}$$

Now let us employ electrostatics. The potential must satisfy Poisson's equation

$$\nabla^2\Phi(\vec{r}) = 4\pi\rho \equiv 4\pi e\,\delta n$$

wher ρ - charge density induced on the neutral background by the probe charge and

$\delta n = n(r) - n_0$ - change of electron density

We assumed that dielectric permeability $\varepsilon = 1$

Thus we can write

$$\nabla^2\Phi = 4\pi e \cdot \left\{ \frac{\left[2m(\mu - e\Phi(r))\right]^{3/2} - (2m\mu)^{3/2}}{3\pi^2\hbar^3} \right\} \quad (*)$$

This is the Thomas - Fermi equation first obtained in the theory of electron density in atoms.

In general this Eq. can be solved only numerically

If we assume that the potential $e\Phi$ is smaller than the Fermi energy $e\Phi \ll \mu$ and expand the RHS of the TF equation (*)

$$\delta n = \frac{3}{2}\frac{e\Phi(r)\,n_0}{\mu}$$

$$\nabla^2\Phi(r) = \frac{1}{l_{TF}^2}\Phi(r) \quad (**)$$

TF screening length $\quad l_{TF} = \frac{\mu^{1/2}}{\sqrt{6\pi}\,e n^{1/2}} = \frac{1}{2}\left(\frac{\pi}{3}\right)^{1/6}\frac{\hbar}{e m^{1/2}}\, n^{-1/6}$

To find the physical meaning of l_{TF} let us solve Eq. (**) imposing the condition that at small distances we should get the potential of point charge $\Phi(r) \xrightarrow[r\to 0]{} Q/r$

$$\Phi(\vec{r}) = \frac{Q}{r}\exp\left(-r/l_{TF}\right)$$

Coulomb potential is modified by the responce of all surrounding electrons. Now it decays exponentiall at the distance l_{TF} from the source $\to$ it is screened.

Physically when a positive charge is brought into the system electrons will be attracted to it forming a negatively charged cloud. If the external charge is negative it will repel other electrons and only positive lattice background will remain.

We see that in metal an electron is surrounded by the screening cloud of other electrons.
Any force applied to it will have to accelerate the whole cloud. Therefore the electron will behave as if it had a larger effective mass m^* than in vacuum.
Instead of being a particle it acquires a finite size of the cloud. This complex structure

electron + cloud = quasiparticle.

Electron is dressed. Logically the electron without cloud is called bare particle.

These quasiparticles interact with each other trough short-range screened potential. This is a great advantage because such interactions can be considered as perturbations. There exists a well-established technology of how to construct such perturbation theory and we will consider it futher in this course.

2. Propagation function in a One-body Quantum theory.

The concept of quasiparticles would not be useful without a general technology which allows to calculate spectrum of quasiparticles, their effective masses and interaction potentials.

Such a general mathematical apparatus is based on the famous Green's functions and Feynman diagrams

Let us start from the case of a single quantum particle.

Propagator: Definition and properties

1. Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi = H\psi \qquad (*)$$

The solution can be written in the following general form

$$\psi(r, t \geq t') = \int dr'\, G(r, t, r', t')\, \psi(r', t') \qquad (**)$$

The kernel $G(r, t, r', t')$ describes the propagation of the ψ-wave function from (r', t') to (r, t). G is called the propagation function. In order to satisfy the

causality principle

The future does not affect the past

$$\left\{ \begin{array}{l} G(r, t, r', t') = 0 \\ t < t' \end{array} \right. \Rightarrow G(r, t, r', t') \equiv G(r, t, r', t')\,\Theta(t - t')$$

Substituting $(**)$ into $(*)$ we get the Eq. for propagator

$$\left(i\hbar \frac{\partial}{\partial t} - H(r)\right) G(r, t, r', t') = i\hbar\, \delta(r - r')\, \delta(t - t') \qquad (***)$$

The RHS of $(***)$ takes into account $\boxed{\frac{\partial}{\partial t}\Theta(t - t') = \delta(t - t')}$

and the initial cond. $\boxed{G(r, t' + 0, r', t') = \delta(r - r')}$

From $(***)$ we see that the propagator is a Green's function of the Schrödinger equation.

Since the solution vanishes at $t > t'$ it is called the retarded Green's

It is easy to calculate the propagator of a free particle.
Let us consider the one-dimension case $\vec{r} = x$

(2)

The Fourier transform of Eq. (***)

$$\left(\hbar\omega - \frac{(\hbar k)^2}{2m}\right) G(k,\omega) = i\hbar$$

$$G(k,\omega) = \frac{i\hbar}{\hbar\omega - \frac{\hbar^2 k^2}{2m}}$$ — but this is not the correct expression yet.

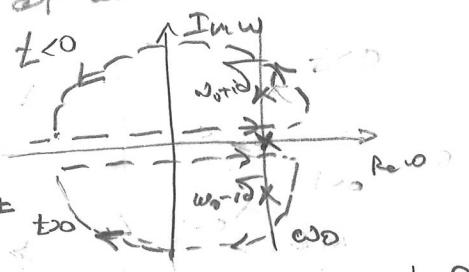
Indeed in real space we have

$$G(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{ikx - i\omega t} G(k,\omega) \quad (***)$$

The integral over ω is convenient to take using complex analysis

$$\oint_C \frac{d\omega}{2\pi} e^{-i\omega t} G(\omega,k) = \pm 2\pi i \sum_{\omega_0} \mathrm{Res}\, G(k,\omega)$$ - the sign depends on direction

where the sum is taken over the residues of all the poles of $G(\omega,k)$ at ω



Important

Since the integral (***) contains $e^{-i\omega t} = e^{-i\mathrm{Re}\,\omega t} e^{\mathrm{Im}\,\omega t}$ we must close the contour in the upper half-plane $\mathrm{Im}\,\omega > 0$ at $t<0$ and in the lower $\mathrm{Im}\,\omega < 0$ if $t>0$.

But the only pole lies exactly on the integration contour

$$\omega_0 = \frac{\hbar^2 k^2}{2m}$$

We can add infinitesimal imaginary part $\omega \pm i\delta$ dramatically changes the answer.

$$G^R(k,\omega) = \frac{i\hbar}{\hbar\omega - \frac{(\hbar k)^2}{2m} + i\delta}$$

For example if we put $\omega \to \omega + i0$ the pole shifts below the real axis. At $t<0$ the contour is in the upper half-plane and does not contain singularities $\Rightarrow$ $\boxed{G^R(t<0) \equiv 0}$

On the other hand, at $t>0$ the contour is in the lower half-plane

$$G^R(x,t) = +\Theta(t) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx - i\frac{\hbar^2 k^2 t}{2m}} =$$ - Retarded function

$$+ \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{im(x-x')^2}{2\hbar t}\right) \Theta(t)$$

$$\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

we can put $\omega \to \omega - i0$ and displace the pole in the upper half-plane and obtain advanced Green's function $G^A = \frac{1}{\omega - \frac{\hbar k^2}{2m} - i0}$

$$G^A = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(\frac{im(x-x')^2}{2\hbar t}\right) \Theta(-t)$$

Perturbation theory for the Propagator

Apart from a few exactly solvable cases the only way to find the propogator is to us perturbation theory

Let us rewrite Schrödinger equation in

Dirac's notation $i\hbar \frac{\partial}{\partial t} |\Phi(t)\rangle = H(t) |\Phi(t)\rangle$ (*)

Assume that Hamiltonian is time-independent

Then formal solution of Eq. (*) can be written as

$$|\Phi(t)\rangle = e^{-i\frac{Ht}{\hbar}} |\Phi(0)\rangle$$

Evolution operator $U = \exp(-iHt/\hbar)$

What if H is time-dependent?

Can we write U $\exp\left(-\frac{i}{\hbar}\int_0^t H\,dt\right)$? No $[H(t_1), H(t_2)] \neq 0$

But we can construct $U(t)$ using iterations. Let us rewrite Schrödinger equation as follows

$$|\Phi(t)\rangle = |\Phi(0)\rangle - \frac{i}{\hbar}\int_0^t H(t')|\Phi(t')\rangle\,dt'$$

Then we get

$$U(t) = I + \left(\frac{-i}{\hbar}\right)\int_0^t dt_1' H(t_1') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1' \int_0^{t_1'} dt_2' H(t_1') H(t_2') +$$

$$\left(\frac{-i}{\hbar}\right)^3 \int_0^t dt_1' \int_0^{t_1'} dt_2' \int_0^{t_2'} dt_3' H(t_1') H(t_2') H(t_3') + \dots$$

In this expression the opepators $H(t)$ are time-ordered so that operators with larger time argument always stays on the left.

We can introduce the time-ordering operator

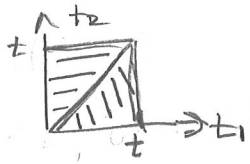
$$T[A(t_A)B(t_B)C(t_C)] = \begin{cases} A(t_A)B(t_B)C(t_C) & \text{if } t_A > t_B > t_C \\ B(t_B)A(t_A)C(t_C) & \text{if } t_B > t_A > t_C \\ A(t_A)C(t_C)B(t_B) & \text{if } t_A > t_C > t_B \\ \dots \end{cases}$$

This operator allows to present the evolution operator in a short form

$$U(t) = T\exp\left(-\frac{i}{\hbar}\int_0^t dt' H(t')\right)$$

Let us check this! Expand the exponent take n-th term

$$\frac{1}{2!}T\left[\int_0^t dt_1 \int_0^t dt_2\, H(t_1)H(t_2)\right] =$$



$$= \frac{1}{2!}\left[\int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_1)H(t_2) + \int_0^t dt_2 \int_0^{t_2} dt_1\, H(t_2)H(t_1)\right] =$$

$$= \int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_1)H(t_2)$$

The same procedure can be done with other terms.

If we evolve from t' then $S(t,t') = T\exp\left(-\frac{i}{\hbar}\int_{t'}^t d\tau H(\tau)\right)$ (*)

Using the evolution operator we can construct propagator. For this let us consider wave function in coordinate representation which can be derived from the Dirac's notation

$$\psi(x,t) = \langle x|\Phi(t)\rangle = \langle x|S(t,t')|\Phi(t')\rangle =$$

$$= \int dx' \langle x|S(t,t')|x'\rangle\langle x'|\Phi(t')\rangle = \int dx' \langle x|S(t,t')|x'\rangle\, \psi(x',t')$$

We see that

$$G(x,t,x',t') = \langle x|S(t,t')|x'\rangle \quad (**)$$

Using the expansion of the evolution operator (*) it is possible to construct perturbation theory and then find the propagator (**)

The idea is to split Hamiltonian into two parts

$$H(t) = H_0 + W(t) \leftarrow \text{small perturbation.}$$

It would be nice to get rid of the large time-independent part H_0 from the evolution operator.

For this purpose we can employ <u>interaction representation</u>.

In this case both operators and state vectors are time-dependent but the evolution of operators is governed by the unperturbed Hamiltonian H_0

$$A(t) = \exp(iH_0t/\hbar)\, A_S \exp(-iH_0t/\hbar)$$

$$i\hbar\frac{d}{dt}A(t) = [A(t), H_0] + i\hbar\cancel{\frac{\partial}{\partial t}A}$$

Operators

$$|\Phi(t)\rangle = \exp(iH_0t/\hbar)\,|\Phi(t)\rangle_S$$ – state vector

$$i\hbar\frac{\partial}{\partial t}|\Phi(t)\rangle = W(t)\,\Phi(t)$$ – obey the equation

Here the perturbation operator is also in the interaction representation

$$W(t) = \exp\left(\frac{iH_0 t}{\hbar}\right) W_S^{(H)} \exp\left(-\frac{iH_0 t}{\hbar}\right)$$

The evolution operator is given by

$$S(t,t') = T \exp\left(-\frac{i}{\hbar}\int_{t'}^{t} d\tau\, W(\tau)\right)$$

It depends only on the perturbation!

<u>Example</u>

Let us consider a particle interacting with external potential

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + V(x,t) \qquad H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}$$

$$W = V(x,t)$$

The propagator is

$$G(x,t,x',t') = \langle x,t | T \exp\left(-\frac{i}{\hbar}\int_{t'}^{t} W\, d\tau\right) | x',t'\rangle =$$

$$= \langle x,t | 1 - \frac{i}{\hbar}\int_{t'}^{t} W\, d\tau + \dots | x',t'\rangle = \langle x,t | x',t'\rangle - \frac{i}{\hbar}\int_{t'}^{t} \langle x,t | W | x',t'\rangle\, d\tau =$$

$$= G_0(x,t,x',t') - \frac{i}{\hbar}\int_{t'}^{t} \langle x,t | x'',\tau\rangle \langle x'',\tau | W | x''',\tau\rangle \langle x''',\tau | x',t'\rangle\, dx''\, dx'''\, d\tau$$

$$= G_0(x,t,x',t') + \int_{t'}^{t} G_0(x,t,x'',\tau)\left(-\frac{i}{\hbar}V(x'',\tau)\right) G_0(x'',\tau,x',t')\, dx''\, d\tau + \dots$$

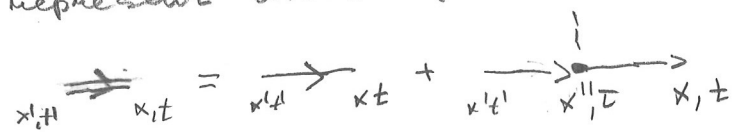
Where we used that

$$G_0(x,t,x',t') = \langle x,t | x',t'\rangle$$

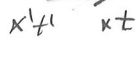
$$\langle x'',\tau | W(\tau) | x''',\tau\rangle = V(x'',t)\,\delta(x''-x''')$$

We can represent above expression as follows

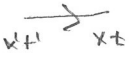
<u>simplest Feynman diagram</u>



The Feynman rules: correspondence between graphic elements and algebr. expressio



$G_0(x,t,x',t')$ Propagator



$G_0(x,t,x',t')$ Free (unperturbed) propagator

$-\frac{i}{\hbar}V(x,t)$ External potential

The integration over all intermediate coordinates and times implied

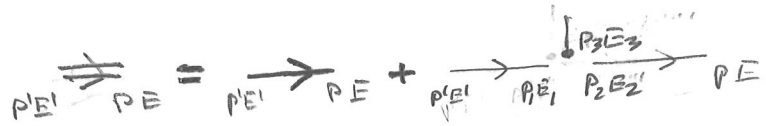
We can use different representations for propagator. For example momentum and frequency (energy)

$$G(p, E, p', E') = \int dx\, dx'\, dt\, dt'\, \exp\left[i(px - p'x') - i(Et - E't')\right] G(x, t, x', t')$$

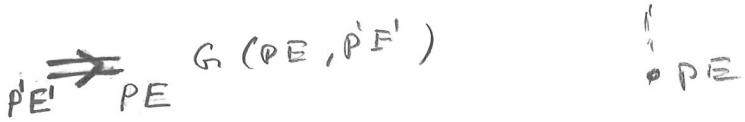
and

$$V(p, E) = \int dx\, dt\, V(x, t)\, \exp\left(-\frac{i}{\hbar}(px - Et)\right)$$

One can derive the following rules



$p_2 = p_1 + p_3$
$E_2 = E_1 + E_3$



$G(pE, p'E')$

$-\frac{i}{\hbar} V(p, E)$

$G_0(pE, p'E')$

Integration over all intermediate momenta/energies is implied taking into account conservation laws.

Home task

6

$$i\hbar\, G^{(2)} = G^0(x', t', x'', \tau)\, V(x'', \tau)\, G_0(x'', \tau, x, t) =$$

$$\int dp'\, dp''\, dp'''\, dp''''\, dE'\, dE_\tau\, dE''_\tau\, dE'''_\tau\, G^0(p', E', p'', E_\tau)\, V(p''', E'_\tau)\, G_0(p'''', E''_\tau, p, E) \cdot$$

$$\cdot \exp\left[i(p'x' - p''x'') - i(E't' - E_\tau \tau)\right] \cdot \exp(ip'''x'' - iE'_\tau \tau) \cdot$$

$$\cdot \exp\left[i(p''''x'' - px) - i(E''_\tau \tau - Et)\right] d\tau\, dx'' =$$

$$\int dp'\, dp''\, dp'''\, dE'\, dE_\tau\, dE'_\tau\, dE''_\tau\, G_0(p'E', p''E_\tau)\, V(p''', E'_\tau)\, G_0(p'''', E''_\tau, p, E)\, \delta(p'''' + p''' - p'')\, \delta(E_\tau - E''_\tau - E'_\tau)$$

$$\times \exp\left[i(p'x' - px) + i(Et - E't')\right] =$$

$$\int dp\, dE\, dp'\, dE'\, \exp\left[i(p'x' - px) + i(Et - E't')\right] G_0(p'E', p''E'')\, V(p'' - p''', E'' - E''')\, G(p''', E''', p, E)$$