Lecture 6

Operations with diagrams. Self energy. Dyson's equation.

The reason why GF are so convenient is that the any part of the diagram can be calculated independently of the structure of the diagram as a whole.

It allows partial summation of diagram series.

The idea is simple. Suppose we have a diagram

infinite subset which describes corrections to the inner line

Due to the fact that there is no explicit dependence of the algebraic expressions on the order of the diagram we can forget about everything that lies beyond these interaction points and concentrate on the inside of the graph.

The sum of all corrections there should transform the thin line (unperturbed GF) to the exact GF → bold line in the same way like the whole series gives an exact propagator

Thus we have partially summed the diagram series!

But there is no guarantee that such approach would work. For example in Superconductivity perturbation series does not converge. It means that superconductivity is not described by the perturbation theory which starts from the non-interacting system. But in many cases the partial summation gives correct results.

To approach it systematically let's introduce some definitions.

Self energy is the part of a diagram connected to the rest of it only by two particle lines

Examples are:

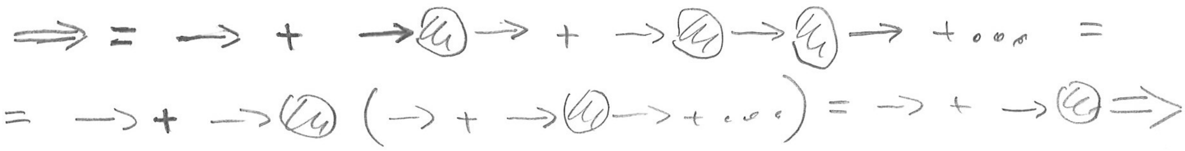
The irreducible (proper) self energy part is the part that can not be separated by breaking one particle line. (2)

Ex: 

The proper self energy or mass operator is the sum of all possible irreducible self-energy parts

$-i\Sigma =$ 

The series for G^F can be drawn as follows



$$iG = iG^0 + iG^0 \Sigma G^0 + iG^0 \Sigma G^0 \Sigma G^0 + \dots =$$

Separating the free propagator we get the Dyson Equation

$$G = G^0 + G^0 \Sigma G$$ – this is a symbolic form

In coordinates

$$G(x,x') = G^0(x,x') + \int dx' \int dx'' G^0(x,x'') \Sigma(x'',x''') G(x'''x')$$

In momentum space it transforms to algebraic expression

$$G(P) = G^0(P) \Sigma(P) G(P)$$

Since $(G^0)^{-1} = \omega - \varepsilon_0(p) + \mu$

$$G(p,\omega) = \left[(G^0(p,\omega))^{-1} - \Sigma(p,\omega) \right]^{-1} = \frac{1}{\omega - \varepsilon(p) + \mu - \Sigma(p,\omega)}$$

Symbolically this can be written as

$$G = \left(i\frac{\partial}{\partial t} - \varepsilon - \hat{\Sigma} \right)^{-1}$$ – it holds even when $\hat{\Sigma}$ is non-diagonal i.e. in the non homogeneous case

It has to be checked that partial summation does not break the general analytical properties.

E.g. from KL-representation we have seen that

$$\text{sign Im } G(p,\omega) = -\text{sign }\omega$$

$$\text{sign Im } \Sigma(p,\omega) = -\text{sign }\omega$$

Thus the calculation of G reduces to that of Σ which requires less diagrams

Quasiparticle spectrum is determined by the poles

$$G^{-1}(\varepsilon-\mu, p) = 0 \,. \qquad \text{since} \quad (G^0)^{-1}(\varepsilon-\mu,p) = \varepsilon(p) - \frac{p^2}{2m} = \Sigma(\varepsilon-\mu,p)$$

Quasiparticle energy $\quad \varepsilon(p) = \frac{p^2}{2m} + \Sigma(\varepsilon-\mu, p)$

Example: electron gas with Coulomb interaction.

neglect connections to the particle lines

1) The first term is the Hartree one. It is compensated by the positive background

2) The second term is

$$-i\Sigma(p) = \int -iV(q)\; i\,G^0(p-q)\frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3} = \int V(q)\, G^0(p-q)\frac{d^3q}{(2\pi)^3}\frac{d\omega}{2\pi}$$

$$G^0 = \frac{1}{\omega - \varepsilon^0(p) + \mu + i\delta\,\mathrm{sgn}\,\omega}$$

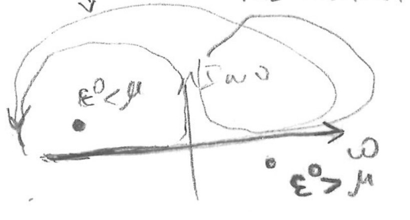
$$\int G^0(\omega, p)\frac{d\omega}{2\pi} = i\,\Theta(p - p_F)$$

$$= i\int V(\vec q)\,\Theta(|p-q| - k_F)\frac{d^3q}{(2\pi)^3} =$$

$$= i\,4\pi e^2 \int_{|\vec q|<k_F} \frac{d^3q}{|\vec p - \vec q|^2}\frac{d^3q}{(2\pi)^3} \qquad V(q) = \frac{4\pi e^2}{|q|^2}$$

! $\lim\limits_{t\to -0} \int G^0(\omega, p) e^{-i\omega t}\frac{d\omega}{2\pi}$

should choose this contour



Finally the excitation spectrum

$$\varepsilon(p) = \frac{p^2}{2m} - 4\pi e^2 \int_{|q|<k_F} \frac{d^3q}{|\vec p - \vec q|^2}\frac{d^3q}{(2\pi)^3}$$

$$m^* = \hbar^2 k_F \left.\frac{\partial \varepsilon}{\partial p}\right|_{k_F}$$

we have already obtained this expression when calculated the energy of interacting gas. There we had problems since such quasiparticles have $m^* = 0$. Now we see from where the problem comes, We neglected important terms in pert. series!

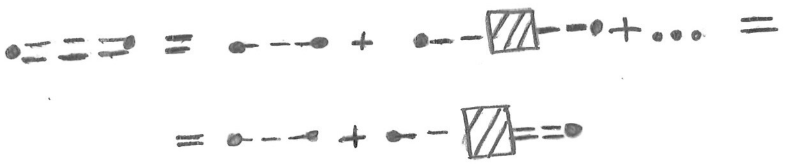
In particular to heal the unphysical answer we can consider another class of diagrams called <u>polarization insertions</u>. They determine renormalization of interactions between particles.

For example  <u>The polarization insertion</u> is called the part of the diagram that is connected to the rest of it only by two interaction lines. <u>The irreducible polarization insertion</u> is one that cannot be separated by breaking of a single interaction line. Finally, the <u>polarization operator</u> is a sum of all irreducible polarization insertions.

One can obtain the analogue to Dyson's equation

(4)

$$U_{eff}(P) = U(P) + U(P)\Pi(P)\,U_{eff}(P)$$



We can introduce the <u>generalized dielectric function</u> $\varkappa$

$$U_{eff}(p,\omega) \equiv \frac{U(p,\omega)}{\varkappa(p,\omega)} = \frac{U(p,\omega)}{1 - U(p,\omega)\Pi(p,\omega)}$$

$\varkappa$ describes how the many-body environment changes interaction between particles.

Example: <u>Screening of Coulomb interaction</u>

The Thomas-Fermi screening length that we derived using some assumptions that were physically correct but not rigorously justified. Now we can obtain this result rigorously and understand what perturbation terms we keep and what we through out.

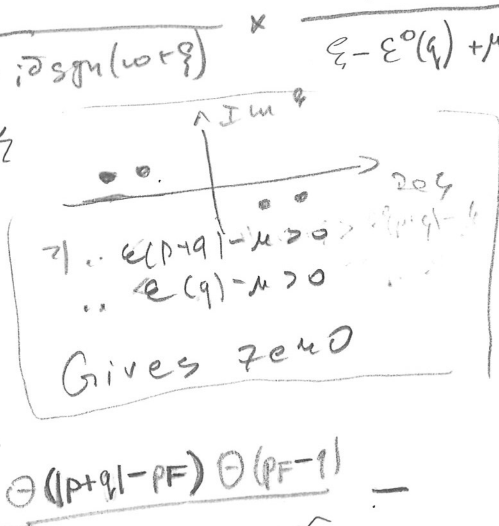
Let us take again only the lowest-order term in the polarization operator

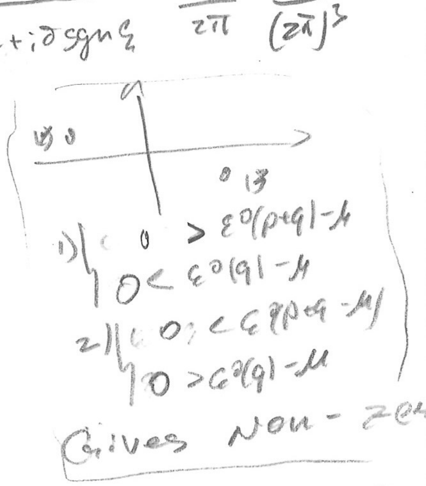
$$i\Pi(P) = $$  - <u>Random phase approximation</u>

$$i\Pi(P) = \underset{\text{spin}}{2}\,\underset{\text{loop}}{(-1)}\cdot i\cdot i\int G^0(P+q)\,G^0(q)\,\frac{d^4q}{(2\pi)^4} =$$

$$i\Pi(p,\omega) = 2\int \frac{1}{(\omega+\xi) - \varepsilon^0(p+q) + \mu + i\delta\,\mathrm{sgn}(\omega+\xi)} \times \frac{1}{\xi - \varepsilon^0(q) + \mu + i\delta\,\mathrm{sgn}\,\xi}\,\frac{d\xi}{2\pi}\,\frac{d^3q}{(2\pi)^3}$$

Integration by frequency ξ





Result:

$$i\Pi(p,\omega) = \frac{2i}{(2\pi)^3}\int d^3q \left[\frac{\Theta(|p+q|-p_F)\,\Theta(p_F-q)}{\omega+\omega_q-\omega_{p+q}+i\delta} - \frac{\Theta(p_F-|p+q|)\,\Theta(q-k_F)}{\omega+\omega_q-\omega_{p+q}-i\delta}\right]$$

$$\omega_{pq} = \frac{(p+q)^2 - p^2}{2m} = \frac{1}{m}\left(p\cdot q + \frac{1}{2}p^2\right)$$

(5) $$\Pi(p,\omega) = \frac{2}{(2\pi)^3}\int d^3q\left[\frac{\Theta(|p+q|-p_F)\,\Theta(p_F-q)}{\omega-\omega_{pq}+i\delta} - \frac{\Theta(p_F-|p+q|)\,\Theta(q-k_F)}{\omega-\omega_{pq}-i\delta}\right]$$

$$q' = -(p+q)$$

$$\Pi(p,\omega) = \frac{2}{(2\pi)^3}\int d^3q\,\Theta(|p+q|-p_F)\,\Theta(p_F-q)\left[\frac{1}{\omega-\omega_{pq}+i\delta} - \frac{1}{\omega+\omega_{pq}-i\delta}\right]$$

a) $\omega = 0$, $p \neq 0$

$$\Pi(p,0) = -\frac{4}{(2\pi)^3}\int d^3q\,\Theta(|p+q|-p_F)\,\Theta(p_F-q)\,\frac{1}{(\omega_{pq}-i\delta)}$$

$$\mathrm{Im}\,\Pi(p,0) = \frac{4}{(2\pi)^3}\int d^3q\,\Theta(|p+q|-p_F)\,\Theta(p_F-q)\,\delta\left((p+q)^2-q^2\right) = 0$$

$$\mathrm{Re}\,\Pi(p,0) = -\frac{4}{(2\pi)^3}\int d^3q\,\Theta(|p+q|-p_F)\,\Theta(p_F-q)\,\frac{1}{\omega_{pq}} =$$

$$= -\frac{4}{(2\pi)^3}\int d^3q\left[1-\Theta(p_F-|p+q|)\right]\Theta(p_F-q)\,\frac{1}{\omega_{pq}} =$$

even function of $(p+q/2)$

$$= -\frac{4m}{(2\pi)^3}\int d^3q\,\frac{\Theta(p_F-q)}{p\cdot q + p^2/2} =$$

$$= -\frac{mp_F}{2\pi^2}\left(1 + \frac{p_F^2-p^2/4}{p_F p}\ln\left|\frac{p_F+p/2}{p_F-p/2}\right|\right)$$

1) Long-range screening $p \ll p_F$

$$\Pi_0 \simeq -2N(\mu) = -\frac{mp_F}{\pi^2},$$ where $N(\mu)$ is the D.O.S. on the Fermi surface.

Hence the interaction becomes

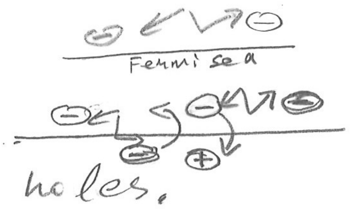
$$U_{eff}(p) = \frac{4\pi^2e^2/p^2}{1+2N(\mu)\,4\pi e^2/p^2} = \frac{4\pi e^2}{p^2+8\pi e^2N(\mu)}$$

The quantity $p_{TF}^2 = 8\pi e^2 N(\mu)$ is the inverse Thomas-Fermi length

The potential in real space has Yukawa form:

$$U_{eff}(r) = \frac{e^2}{r}\exp(-p_{TF}r)$$

From the diagram we see the physical origin of renormalization



Interaction creates virtual pairs of electrons and holes. Each pair is independent since the quantum mechanical phase of each pair is lost when they annihilate. – hence the name RPA.

a) Friedel oscillations

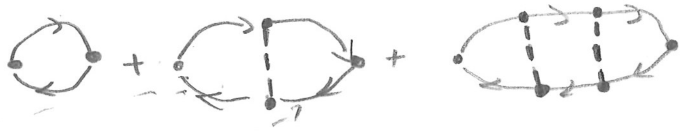
The polarization operator has singularity at $p = 2p_F$. Therefore the effective potential in real space has rapidly oscillating part. This effect is called F.O. or Ruderman-Kittel oscillations

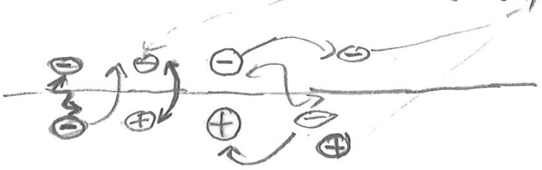
$$V(r) \sim \frac{1}{r^3}\cos(2k_F r)$$ - asymptotic at large distances.

RPA works well if the particle density is high so that the probability that the electron from virtual electron-hole pair will interact with other particles is much larger than they will interact with each other.

In the opposite case when the density is small electron and hole in the virtual pair interact several time before disappearing

Ladder approximation

$i\Pi =$ 



Let us apply the result to calculate effective mass

$$\Sigma(p) = -\int_{|\vec{q}|<k_F} V(\vec{p}-\vec{q}) \frac{d^3q}{(2\pi)^3} = -\int_{|\vec{q}|<k_F} \frac{V_0(\vec{p}-\vec{q})}{\varkappa} \cdot \frac{d^3q}{(2\pi)^3}$$

Consider the simplified case of long-wave screening

$$\varkappa(q) = 1 + \left(\frac{q_0}{q}\right)^2$$

Then

$$\Sigma(p) = -V_0 \int_{|q|<k_F} \frac{2\pi \sin\theta \, d\theta \, q^2 \, dq}{p^2+q^2-2pq\cos\theta + q_0^2} = -\frac{\pi V_0}{p}\int_0^{p_F} q\, dq \ln\left|\frac{p^2+q^2-2pq+q_0^2}{p^2+q^2+2pq+q_0^2}\right|$$

$$\left.\frac{dE}{dp}\right|_{p=p_F} = \frac{p_F}{m_0} + \frac{d\Sigma}{dp}(p=p_F)$$

$$\Sigma = -\frac{\pi V_0}{p}\int_0^{p_F} q\, dq \left[\ln\left((p-q)^2+q_0^2\right) - \ln\left((p+q)^2+q_0^2\right)\right] =$$

$$= -\frac{\pi V_0}{p}\left[\int_{-p}^{p_F-p} (y+p)\ln(y^2+q_0^2)\, dy - \int_p^{p_F+p} (y-p)\ln(y^2+q_0^2)\, dy\right]$$

$$\left.\frac{d\Sigma}{dp}\right|_{p=p_F} = -\frac{\pi V_0}{p_F}(2p_F)\cdot \ln\left(\frac{q_0}{p_F}\right) = 2\pi V_0 \ln\left(\frac{p_F}{q_0}\right)$$

leading terms in $\ln\left(\frac{p_F}{q_0}\right)$

Effective mass is finite

$$\frac{1}{m^*} = \frac{1}{m_0} + \frac{2\pi V_0}{p_F}\ln\left(\frac{p_F}{q_0}\right)$$

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b) $p \to 0$, $\omega \neq 0$

$$\left(1-\Theta(\cancel{p_F}-|p+q|)\right)\Theta(p_F-q)$$

$$\operatorname{Re}\Pi(0,\omega) = \frac{2}{(2\pi)^3}\int d^3q \; \frac{2\omega_{pq}}{\omega^2}\;\Theta(|p+q|-p_F)\,\Theta(p_F-q) =$$

$$= \frac{2}{(2\pi)^3}\frac{1}{m\omega^2}\int d^3q\,\underbrace{(2p\cdot q+p^2)}_{(p+q)^2-q^2}\,\Theta(|p+q|-p_F)\,\Theta(p_F-q) =$$

$$= \frac{2}{(2\pi)^3}\frac{1}{m\omega^2}\int d^3q'\left((q+p/2)^2-(q-p/2)^2\right)\Theta(|q+p/2|-p_F)\,\Theta(p_F-|q-p/2|)$$

$$= \frac{2}{(2\pi)^3}\frac{1}{m\omega^2}\int d^3q'\left((q+p/2)^2-(q-p/2)^2\right)\Theta(p_F-|q-p/2|) =$$

$$= \frac{2}{(m\omega^2)}p^2\int\frac{d^3q}{(2\pi)^3}\,\Theta(p_F-q) = \frac{2p^2}{m\omega^2}\cdot\frac{4\pi p_F^3}{3(2\pi)^3} = \frac{p^2p_F^2}{3m\omega^2\pi^2}$$

Dielectric function

$$\varepsilon = 1-\frac{4\pi e^2}{\cancel{p^2}}\cdot\frac{\cancel{p^2}\,p_F^2}{3m\omega^2\pi^2} = 1-\frac{4e^2p_F^2}{3\pi m\,\omega^2} = 1-\frac{4}{3\pi}\frac{e^2\omega_0}{\omega^2} =$$

$$= 1-\frac{4}{3\pi}\left(\frac{\omega_0}{\omega}\right)^2 r_s\alpha$$

where $\alpha = \left(\frac{4}{9\pi}\right)^{1/3}$

$\hbar\omega_0 = \frac{p_F^2}{m}$ $\qquad e^2 = \frac{\hbar^2}{ma_0}$

$(r_0k_F)^3 = \frac{9\pi}{4}$ $\qquad e^2/\omega_0 = \frac{me^2}{p_F^2} = \frac{1}{a_0p_F} = \frac{r_0}{a_0}\left(\frac{4}{9\pi}\right)^{1/3}$

$a_0 = \frac{\hbar^2}{me^2}$

$$\omega_p^2 = \frac{4}{3\pi}\frac{e^2}{m}p_F^2 = \frac{4\pi n e^2}{m}$$ Plasma frequency

Longitudinal plasma waves $\qquad \varepsilon(q,\omega)=0$

$\omega = \omega_p$ $\qquad$ – plasma oscillations degenerate electron gase

c) ω/q is finite, $\omega \to 0$

$\varepsilon(\omega,q)=0$ gives zero sound modes in electron gas with short-range repulsion.

$$\omega^2(k) = \frac{nV(0)}{m}k^2$$

If $V(q) = \frac{4\pi e^2}{q^2}$ it reproduces plasma waves

$$\int_0^{p_F} q\,dq \left(\ln\left(p^2+q^2-2pq+q_0^2\right) - \ln\left(p^2+q^2+2pq+q_0^2\right)\right) =$$

$$= \int_0^{p_F} q\,dq \left[\ln\left((p-q)^2+q_0^2\right) - \ln\left((p+q)^2+q_0^2\right)\right] =$$

$y = q - p \quad y = q + p$

$q = y + p \quad q = y - p$

$$= \int_{-p}^{p_F-p} (y+p) \ln\left(y^2+q_0^2\right) dy - \int_{p}^{p_F+p} (y-p) \ln\left(y^2+q_0^2\right) dy$$

$$\frac{d}{dp} = (y+p)\ln\left(y^2+q_0^2\right)\left(y = p_F - p_F\right) - (y+p_F)\ln\left(y^2+q_0^2\right)\left(y=-p_F\right) -$$

$$- (y-p_F)\ln\left(y^2+q_0^2\right)\left(y = p_F+p_F\right) + (y-p_F)\ln\left(y^2+q_0^2\right)\left(y=p_F\right)$$

$$= p_F \ln\left(q_0^2/p_F^2\right) = 2p_F \ln\left(q_0/p_F\right)$$

leading terms.

To page 7 effective mass calculation