Lection 7. Two particle Green's functions.

T-ordered product of 4 ψ-operators

$$K_{34,12} = \langle T \psi_3 \psi_4 \psi_1^+ \psi_2^+ \rangle$$

is a two-particle GF.

To construct perturbation theory we introduce interaction representation and obtain

$$K_{34,12} = \frac{1}{\langle S \rangle} \langle T \psi_{I3} \psi_{I4} \psi_{I1}^+ \psi_{I2}^+ S \rangle$$

To the zeroth order this average splits to the sum of two products of free propagators

$$K^0_{34,12} = G^0_{31} G^0_{42} - G^0_{32} G^0_{41}$$

Further it is convenient to introduce momentum represent. In the spatially homogeneous case K depends on the coordinate differencies $x_3 - x_2$, $x_4 - x_2$, $x_1 - x_2$ hence the Fourier transform contains $\delta(p_1 + p_2 - p_3 - p_4)$.

Indeed since

$$p_3 x_3 + p_4 x_4 - p_1 x_1 - p_2 x_2 = p_3(x_3 - x_2) + p_4(x_4 - x_2) - p_1(x_1 - x_2) -$$
$$- x_2(p_1 + p_2 - p_3 - p_4)$$

Then $$\int K_{34,12} \exp(i(p_3 x_3 + p_4 x_4 - p_1 x_1 - p_2 x_2))\, d^4 x_{1234} =$$
$$= (2\pi)^4 \delta(p_3 + p_4 - p_1 - p_2) \underbrace{K_{\gamma\delta,\alpha\beta}(p_3 p_4, p_1 p_2)}$$

2-particle GF in momentum represent
it's arguments satisfy $p_1 + p_2 = p_3 + p_4$

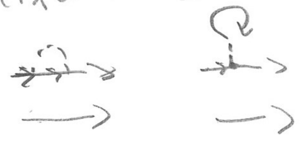
In zeroth approximation

$$K^0_{\alpha_3\alpha_4,\alpha_1\alpha_2}(p_3, p_4; p_1, p_2) = (2\pi)^4 \Big[\delta(p_1 - p_3)\, G^0_{\alpha_3\alpha_1}(p_1) G^0_{\alpha_4\alpha_2}(p_2) -$$
$$\delta(p_1 - p_4)\, G^0_{\alpha_3\alpha_2}(p_2) G^0_{\alpha_4\alpha_1}(p_1)\Big]$$

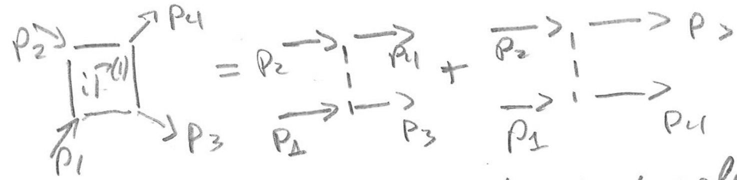
$\xrightarrow{p_1 = p_3}$ $\qquad$ $\xrightarrow{p_1 = p_4}$

$\xrightarrow[p_2 = p_4]{}$ $\qquad$ $\xrightarrow[p_2 = p_3]{}$

To the higher orders K has several qualitatively different contributions

a) 

Modifications of separate lines

b) 

The diagrams which do not split to separate parts

The general expression looks like

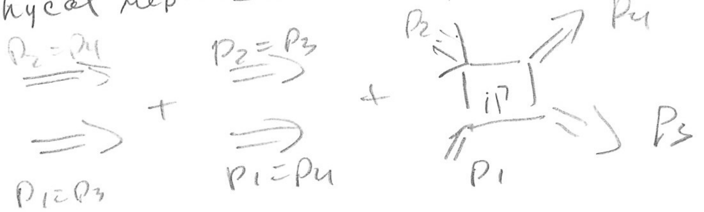
$$K_{\alpha_3\alpha_4,\alpha_1\alpha_2}(P_3P_4,P_1P_2)=(2\pi)^4\Big[\delta(P_1-P_3)\,G_{\alpha_3\alpha_1}(P_1)\,G_{\alpha_4\alpha_2}(P_2)\,-$$
$$-\,\delta(P_1-P_4)\,G_{\alpha_3\alpha_2}(P_2)\,G_{\alpha_4\alpha_1}(P_1)\Big]\,+$$
$$+\,G_{\alpha_3\beta_3}(P_3)\,G_{\alpha_4\beta_4}(P_4)\;i\Gamma_{\beta_3\beta_4\beta_1\beta_2}(P_3P_4,P_1P_2)\,G_{\beta_1\alpha_1}(P_1)\,G_{\beta_2\alpha_2}(P_2)$$

Short notation

$$K(P_3P_4,P_1P_2)=(2\pi)^4\Big[\delta(P_1-P_3)\,G(P_1)G(P_2)-G(P_2)G(P_1)\,\delta(P_1-P_4)\Big]+$$
$$+\,G(P_3)G(P_4)\;i\Gamma(P_3P_4,P_1P_2)\,G(P_1)G(P_2)$$

(spin matrices are implied)

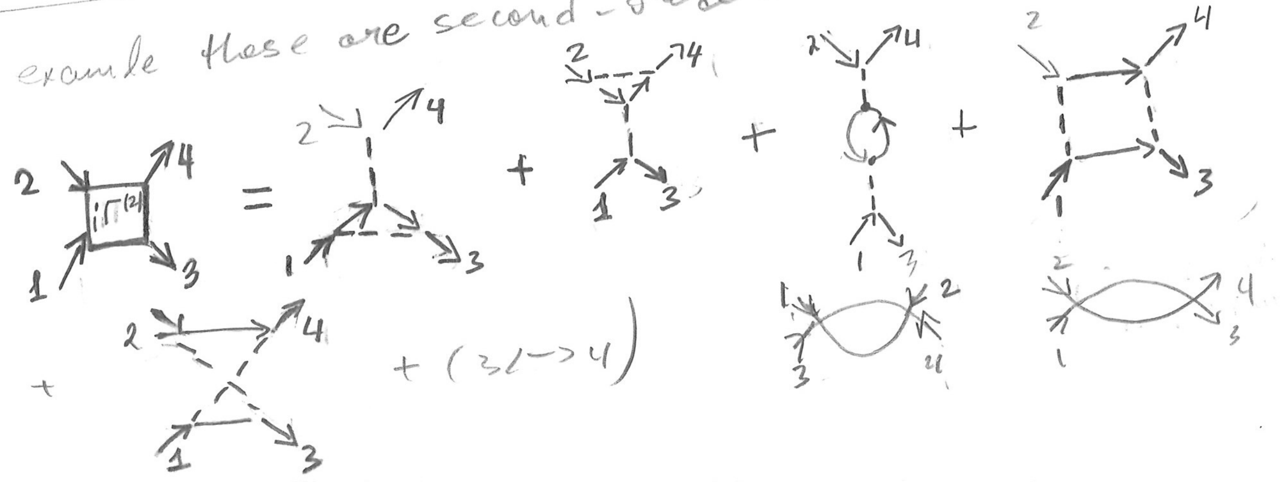
Graphycal representation



$i\Gamma$ is vertex

The calculation of algebraic expressions uses the same F.r. as before and one more in addition: if there is a continuous sequence of solid lines which goes from 1 to 4 and 2 to 3 then the sign of diagram changes.

For examle these are second-order terms in the vertex



Dyson eq. for vertex and self-energy functions

Consider the propagator $G(x_1 x_2) = -i\langle T\psi(x_1)\psi^+(x_2)\rangle$ and calculate the time derivative

$$i\frac{\partial}{\partial t_1} G(x_1 x_2) = \frac{\partial}{\partial t_1}\langle T\psi(x_1)\psi^+(x_2)\rangle = \delta(x_1-x_2) + \langle T\frac{\partial}{\partial t_1}\psi(x_1)\psi^+(x_2)\rangle$$

$$= \delta(x_1-x_2) + i\langle T[H,\psi(x_1)]\psi^+(x_2)\rangle$$

we use the equation of motion for ψ-operator

$$i\frac{\partial}{\partial t}\psi = -[H,\psi] = \left(-\frac{\nabla^2}{2m}-\mu\right)\psi + \int \psi^+(x')\psi(x')\psi(x)\,U(x-x')dx'$$

Then

$$\left(i\frac{\partial}{\partial t_1}+\frac{\nabla_1^2}{2m}+\mu\right) G(x_1 x_2) = \delta(x_1-x_2) - \underbrace{i\int\langle T\psi^+(x_3)U(x_1-x_3)\psi(x_3)\,d^4x_3\,\psi(x_1)\psi^+(x_2)\rangle}_{-i\int K(x_3 x_1, x_3 x_2)\,U_{13}\,d^4x_3}$$

Finally let us transform to the momentum representation

$$\int d^4(x_1-x_2)\exp(ip(x_1-x_2)) \times [\quad]$$

$$K(x_3 x_1, x_3 x_2) = \int K(p_3, p_4, p_1, p_3+p_4-p_1) \times \exp(-ip_3(x_3-x_2) - ip_4(x_1-x_2) + ip_1(x_3-x_2))\frac{d^4p_{3,4}d^4p_1}{(2\pi)^{12}}$$

$(x_3\ x_4\ x_1\ x_2)$

$$U(x_1-x_3) = \int U(Q)\,e^{-i(x_1-x_3)Q}\frac{d^4Q}{(2\pi)^4}$$

overall exponent is

$$\int dx_3\exp(-ip_3(x_3-x_2) - ip_4(x_1-x_2) + ip_1(x_3-x_2) - iQ(x_1-x_3)) \doteq$$

$$= (2\pi)^4\delta(Q-p_1+p_3)\exp(i(x_1-x_2)(p_1-p_3-p_4))$$

Then $\underline{p} = p_3+p_4-p_1$

Finally the p-component

$$\text{F.T.}\int K(x_3x_1, x_3x_2)U_{13}d^4x_3 = \int K(p_3 p_4, p_3+p_4-p, p)\,U(p-p_4)\frac{d^4p_3 d^4p_4}{(2\pi)^8}$$

Substituting definition of vertex function

$$\left((G^0)^{-1}(p) - G^{-1}(p) \equiv \Sigma(p)\right) = U(0)\overset{\text{density}}{n} + i\int U(p-p_1)G(p_1)\frac{d^4p_1}{(2\pi)^4}$$

$$+ \int \Gamma(p_3, p_4, p_3+p_4-p, p)\,G(p_3)\,G(p_4)\,G(p_3+p_4-p)\,U(p-p_4)\frac{d^4p_3 d^4p_4}{(2\pi)^8}$$

$i \quad i \quad i \quad -i$

$$\langle T\psi_3\psi_4\psi_1^+\psi_2^+ S\rangle = -i\int \langle T\psi_3\psi_4\psi_1^+\psi_2^+\psi_5^+\psi_6^+\psi_6\psi_5\rangle\, U_{56}\, dx_5\, dx_6$$

$$\langle T\psi_2\psi_6^+\rangle \longrightarrow \langle T\psi_6\psi_4^+\rangle = \delta,$$

$\alpha_2\alpha_6$ $\quad | U_{56}$ $\quad \alpha_6\alpha_4$

$$= G_{\alpha_2\alpha_5}(p_2)\, G^{(p_4)}_{\alpha_5\alpha_4} G^{(p_1)}_{\alpha_1\alpha_6} G^{(p_3)}_{\alpha_6\alpha_3} U_{56}$$

$p_1 \longrightarrow | \longrightarrow p_3$

$\langle T\psi_1\psi_5^+\rangle\langle T\psi_5\psi_3^+\rangle$

$\alpha_1\alpha_5 \quad \alpha_5\alpha_3$

$$\delta_{\alpha_2\alpha_5}\delta_{\alpha_5\alpha_4}\delta_{\alpha_1\alpha_6}\delta_{\alpha_6\alpha_3} = \delta_{\alpha_2\alpha_4}\delta_{\alpha_1\alpha_3}$$

$$Q = p_3 - p_1$$

Fourier

$$\int K_{34,12}\exp\left(i(p_3x_3 + p_4x_4 - p_1x_1 - p_2x_2)\right) d^4x_{1234} =$$

$$\int k(x_3x_4, x_1x_2)\exp\left(i(p_3x_3 + p_4x_4 - p_1x_1 - p_2x_2)\right) d^4x_{1\ldots 4} =$$

$$= (2\pi)^4\delta(p_3+p_4-p_1-p_2)\, k(p_3p_4, p_1p_2)$$

Inv. Fourier

$$k(x_3,x_4,x_1,x_2) = \int \delta(p_3+p_4-p_1-p_2)\exp\left(-i(p_3x_3+p_4x_4-p_1x_1-p_2x_2)\right)\times$$

$$\times k(p_3,p_4,p_1,p_2)\, d^4p_{1\ldots 4} =$$

$$= \int k(p_3,p_4,p_1,p_3+p_4-p_1)\exp\left(-i(p_3x_3+p_4x_4-p_1x_1-(p_3+p_4-p_1)x_2\right) d^4p_{1,3,4}$$

$$= \int k(p_3,p_4,p_1,p_3+p_4-p_1)\exp\left(ip_3(x_2-x_3)+ip_4(x_2-x_4)+ip_1(x_1-x_2)\right) d^4p_{1,3,4}$$

Dyson

$$k(x_3x_1, x_3x_2) = \int k(p_3,p_4,p_1,p_3+p_4-p_1)\exp\left(ip_3(x_2-x_3)+ip_4(x_2-x_1)+ip_1(x_3-x_2)\right)\frac{d^4p_{1,3,4}}{(2\pi)^{12}}$$

$3\ 4\ 1\ 2$

$$U(x_1-x_3) = \int U(Q)e^{-i(x_1-x_3)Q}\frac{d^4Q}{(2\pi)^4}$$

$$\int k(x_3x_1, x_3x_2)\, U_{13}\, d^4x_3 = \int k(p_3,p_4,p_1,p_3+p_4-p_1)\, U(Q)\times$$

$$\times\exp\left(ip_3(x_2-x_3)+ip_4(x_2-x_1)+ip_1(x_3-x_2)+i(x_3-x_1)Q\right)\frac{d^4p_{1,3,4}\, d^4Q}{(2\pi)^{16}}\, d^4y_3 =$$

$$= \exp\left(ix_3(p_1+Q-p_3)+ip_3x_2+ip_4(x_2-x_1)-ip_1x_2-iQx_1\right) \Rightarrow$$

$$\Rightarrow (2\pi)^4\delta(p_1+Q-p_3)\exp\left(ip_4(x_2-x_1)+ip_3x_2-ip_1x_2-i(p_3-p_1)x_1\right) =$$

$$= (2\pi)^4\delta(p_1+Q-p_3)\exp\left(ix_1(p_1-p_3-p_4)+ix_2(p_4+p_3-p_1)\right) =$$

$$= (2\pi)^4\delta(p_1+Q-p_3)\exp\left(i(x_1-x_2)(p_1-p_3-p_4)\right)$$

$$k(x_3x_1, x_3x_2) = \int k(p_3,p_4,p_1,p_3+p_4-p_1)\, U(p_3-p_1)\exp\left(i(x_1-x_2)(p_1-p_3-p_4)\right)\frac{d^4p_{134}}{(2\pi)^{12}}$$

$$u(x_1-x_3) = \int u(Q)\, \exp(-i\, Q(x_1-x_3))\, \frac{d^4Q}{(2\pi)^4}$$

(3a)

$u \quad)$

$X_3 = \hat{X}_3 \quad X_4 = \hat{X}_1 \quad X_3 = \hat{X}_3 \quad \hat{X}_2 = X_2$

$k(x_3 x_1, x_3, x_2) =$

$$\overset{\psi_1^+}{\psi^+(x_3)}\, \overset{\psi_2^+}{\psi^+(x_2)} \mid \overset{\psi_3}{\psi(x_3)}\, \overset{\psi_4}{\psi(x_1)} = k(3\,1,\ 3\,2) = \psi_3\, \psi_1\, \psi_3^+\, \psi_2^+$$

$$k_{34,12} = \int k(P_3 P_4,\ P_1,\ P_3+P_4-P_1) \times \exp(-i[P_3(X_3-X_2)+P_4(X_1-X_2) -$$

$$- P_1(X_3-X_2)] \frac{d^4p_1 d^4p_3 d^4p_4}{(2\pi)^{12}}$$

$$-P_3(X_3-X_2) - P_4(X_4-X_2) + P_1(X_1-X_2) =$$

$$= -P_3(\hat{X}_3-\hat{X}_2) - P_4(\underline{\underline{\hat{X}_1}}-\hat{X}_2) + P_1(\hat{X}_3 - \hat{X}_2) \quad - i\,Q(\underline{\underline{\hat{X}_1}}-\hat{X}_3)$$

$$= i\hat{X}_3(Q-P_3+P_1) - i\,Q\,\hat{X}_1 + i\,P_3\hat{X}_2 - i\,P_4(\hat{X}_1-\hat{X}_2) - i\,P_1\hat{X}_2$$

$$\delta(Q-P_1+P_3)\ \exp(-i(P_3-P_1)\hat{X}_1 + i P_3\hat{X}_2 - iP_4(\hat{X}_1-\hat{X}_2) - iP_1\hat{X}_2)$$

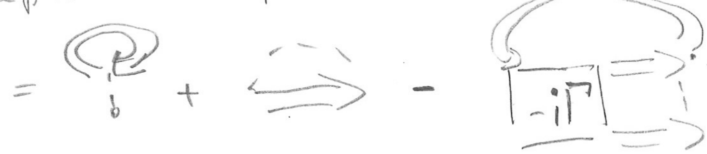
$$k(P_4P_3;\ P_1,\ P_3+P_4-P_1)\ u(Q)\ d^4p_1\, d^4p_3\, d^4p_4\ d^4Q$$

$$\exp: \quad \hat{X}_1(-P_3+P_1-P_4) + \hat{X}_2(P_3+P_4-P_1) = \underbrace{(\hat{X}_1 - \hat{X}_2)(P_1-P_3-P_4)}_{}$$

external Fourier
component

$P =$

Graphical representation

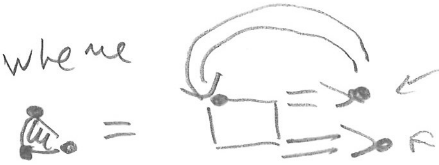
$-i\Sigma =$ 

The first two terms here give the self-consistent Hartree-Fock approximation with bare interaction.

They take into account the interaction of particle with environment and with itself.

The rest (third term) contains renormalization of interaction. It can be represented as follows

where



interaction attached here

It is renormalized interaction vertex

The third terms contains all polarization insertions

The poles of vertex functions determine Bosonic quasiparticle branches

In some particularly interesting cases

$$H_{int} = \frac{1}{2}\int \psi^+_\uparrow(r)\,\psi^+_\downarrow(r)\,\psi_\uparrow(r)\,\psi_\downarrow(r)\,d^3r$$

($\lambda<0$) $\Gamma(\omega)$ has poles in the upper half-plane! which means that in real space the 2-particle function grows in time! This is unstability of the ground state.