Landau Fermi liquid theory

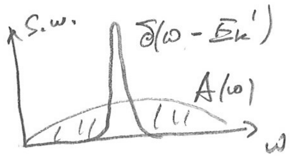
1. Concept of quasiparticles

Consider many-body fermion state $|G\rangle$. Add particle at some time $t=0$ and look at the amplitude of this state at later time

$$G_k(t) = \langle G| e^{iHt} a_k e^{-iHt} a_k^+ |G\rangle = \langle G| a_k(t)\, a_k^+(0) |G\rangle$$

Use Chalen-Lehman representation

$$G_k(t) = \sum_m \langle G| a_k(t) |m\rangle \langle m| a_k^+(0) |G\rangle =$$



$$= \sum_m \underbrace{|\langle G| a_k |m\rangle|^2}_{\text{spectral weight}} e^{-i(E_m - E_g)t}$$

In F.L. state the distribution of spectral weight has peak + continuum.

Therefore $G_k(t) = Z e^{-iE_k' t} + \int \frac{d\omega}{2\pi} A(\omega) e^{-i\omega t}$

Consider physical content of the self-energy

$$G(k,\omega) = \frac{1}{\omega - \varepsilon_k + i\,\mathrm{sgn}\,\omega}$$ – free fermion propagator

$$G(k,\omega) = \frac{1}{\omega - \varepsilon_k - \Sigma_1(k) - \Sigma_2(k,\omega)}$$ – with interactions

1-st order is frequency-independent $\Sigma_1 = \ldots + \ldots = V(0)\sum_{k'} n(k') - \sum_{k'} V(k-k') n_{k'}$

2-nd order $\Sigma_2 =$ [diagram with lines q, $p+q$, p, q, $k-q$]

$$= \int dp\, dq\, d\frac{E_p\, dE_q}{(2\pi)^2} \left((E_p + E_q) - \varepsilon_{p+q} + i\,\mathrm{sgn}\,\varepsilon_{p+q}\right)^{-1} \left(E_p - \varepsilon_p + i\,\mathrm{sgn}\,\varepsilon_p\right)^{-1} \left(\omega - E_q - \varepsilon_{k-q} + i\,\mathrm{sgn}\,\varepsilon_{k-q}\right)$$

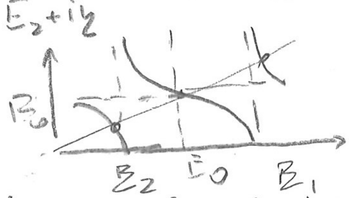
$$= \text{(see calculation in appendix)} \int dp\, dq \frac{\left[\Theta(\varepsilon_{p+q}) - \Theta(\varepsilon_p)\right]\left[1 + \Theta(\varepsilon_{p+q} - \varepsilon_p) - \Theta(\varepsilon_{k-q})\right]}{\omega - \varepsilon_{k-q} + \varepsilon_p - \varepsilon_{p+q} + i\,\mathrm{sgn}\,\omega}$$

$$\Sigma_2(k,\omega) = V_q^2 \left[\quad\right]$$

Σ_2 explicitly depends on ω and has poles at $\omega = \varepsilon_{k-q} + \varepsilon_{p+q} - \varepsilon_p$ and therefore an imaginary part. Consider a simple example

$$\Sigma_2 = \frac{A_1}{\omega - E_1 + i\eta} + \frac{A_2}{\omega - E_2 + i\eta} \longrightarrow G = \frac{1}{\omega - E_0 - \Sigma_2}$$

Poles are at $\omega = E_0 + \Sigma_2$



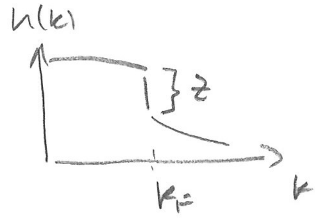
3 poles instead of 1

These poles represent complicated spectrum of excitations in many-body system

Consider realistic case $\omega - \Sigma - E_0 \approx (1 - \Sigma'(E_0))(\omega - E_0) - \underbrace{\Sigma(E_0)}_{\text{neglect}}$

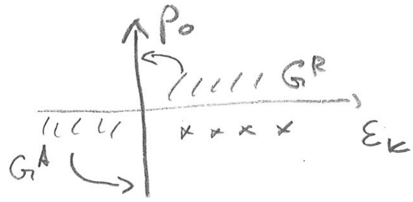
$$\frac{1}{1-\Sigma'_\omega} = 1 - \underbrace{\sum_{p,q} \frac{A_{pq}^2}{(E_0 - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{p-q}))^2}}_{} = Z$$

The strenght of the pole at E_0 is depleted due to uncoherent background



Landau Fermi liquid theory is based on the assumption $G = \frac{Z}{\omega - \varepsilon_p + i0\,\mathrm{sign}\,\omega}$ $Z \neq 0$

$\varepsilon_p = v_F'(p - p_F)$ but $v_F \neq \frac{p_F}{m}$



In the imaginary frequency

$$G = \frac{Z}{ip_0 - \varepsilon_p}$$

Fermi surface survives as a boundary between emty and occupied states

Collective modes smooth perturbations of the vacuum cannot change topology of the fermionic spectrum, that is to destroy the boundary between occupied/empty states.
But CM produce effective fields acting on single particles
The effective fields cannot destroy FS but can localy shift it
Therefore, collective motion of the vacuum is seen by the individual QPs as dynamical modes of the FS.
These boson modes are known as different harmonics of zero-sound

Landau interaction function

Energy $E - E_0 = \sum_{p,\sigma} \varepsilon(p)\,\delta n_{p,\sigma}$ where $\delta n_{p,\sigma}$ - perturbation of distribution function

Interaction between QPs

$$\delta E = \sum_{p\sigma} \frac{\partial E}{\partial n_{p\sigma}} \delta n_{p\sigma} + \frac{1}{2} \sum_{pp'\sigma\sigma'} \underbrace{\left(\frac{\partial^2 E}{\partial n_{p\sigma}\,\partial n_{p'\sigma'}} \right)}_{\frac{1}{V} f(pp'\sigma\sigma')} \delta n_{p\sigma}\,\delta n_{p'\sigma'}$$

$\frac{1}{V} f(pp'\sigma\sigma')$ - Landau function

Renormalizations due to interactions

1) Compressibility: consider s-wave channel $f_0^s = F_0^s / N_0$ Bulk modulus $\varkappa = -V \frac{\partial P}{\partial N} = -\frac{\partial \varepsilon}{\partial n}$

$$\delta \varepsilon^{(2)} = \frac{F_0^s}{2N(0)} \sum_{p,p'} \delta n_p \delta n_{p'} = \frac{1}{2N_0} F_0^s (\delta n)^2$$

Let's introduce molecular field related to the density fluctuation

$$h_{mol} = -\frac{\partial \varepsilon}{\partial n} = -\frac{1}{N_0} f_0^s \delta n = -\frac{1}{N} f_0^s \chi_s \delta \mu \qquad \chi_s = \frac{\partial n}{\partial \mu}$$

Consider energy: Bare $\delta \varepsilon = \delta n \chi_s^0$ FL: $\delta \varepsilon = \delta n \chi_s$

Then $\delta n = \chi_{s0} (\delta \mu + h_{mol}) = \chi_{s0} \delta \mu \left(1 - \frac{1}{N_0} F_0^s \chi_s\right)$

$$\chi_s = \frac{\chi_{s0}}{1 + F_0^s \chi_{s0}/N_0} = \frac{N_0}{1+F_0^s}$$

2) Effective mass: consider p-wave channel

$$j = \int \frac{d^3p}{(2\pi)^3} \nabla_p \varepsilon_p (x,t)\, n_p (x,t) \qquad \varepsilon_{p\sigma} = \varepsilon_p^0 + \int \frac{d^3p}{(2\pi)^3} f(pp') \delta n_{p'} (x,t)$$

$$j = \int \frac{d^3p}{(2\pi)^3} \left(\nabla_p \varepsilon_p^0 \delta n_p + \nabla_p \delta \varepsilon_p n_p^0\right) = \int \frac{d^3p}{(2\pi)^3} \left(\nabla_p \varepsilon_p^0 \delta n_p - \delta \varepsilon_p \nabla_p n_p^0\right) =$$

$$= \int \frac{d^3p}{(2\pi)^3} v_p \left(\delta n - \frac{\partial n_p^0}{\partial \varepsilon_p} \int \frac{d^3p'}{(2\pi)^3} f(pp') \delta n\right)$$

$$-\int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(\frac{\partial n}{\partial \varepsilon_p}\right) f(pp') = N_0 \left(\int \frac{d\Omega}{4\pi} f_1^s \cos^2\theta\, v_F\right) \vec{p}' = \frac{N_0}{3} f_1^s v_F \vec{p}' = \frac{F_1^s \vec{v}_{p'}}{3}$$

$$\vec{j} = \int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(1 + F_1^s/3\right) \delta n_p = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{m^*} \delta n_p$$

$$1/m = 1/m^* \left(1 + F_1^s/3\right)$$

3) Spin susceptibility

The Hartree Fock approximation $f = V_0 \delta_{\alpha\beta} \delta_{\gamma\delta} - V(p-p') \delta_{\alpha\delta} \delta_{\gamma\beta}$

Using relation $\sigma_{\alpha\beta} \sigma_{\gamma\delta} = \delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma}$

We can write in HF approximation

$$f_{\alpha\beta,\gamma\delta} = \left(V(0) - \tfrac{1}{2} J(p-p')\right) \delta_{\alpha\beta} \delta_{\gamma\delta} - \tfrac{1}{2} J(p-p') \vec{\sigma}_{\alpha\beta} \vec{\sigma}_{\gamma\delta}$$

In general $f_{\alpha\beta,\gamma\delta}(p,p') = f^s(p,p') \delta_{\alpha\beta} \delta_{\gamma\delta} + f^a(p p') \vec{\sigma}_{\alpha\beta} \vec{\sigma}_{\gamma\delta}$

The system has global SU(2) symmetry

If we have spin quantization axis $f^s = \frac{1}{2}(f_{\uparrow\uparrow} + f_{\uparrow\downarrow}) \quad f^a = \frac{1}{2}(f_{\uparrow\uparrow} - f_{\uparrow\downarrow})$

Consider s-wave channel $f_0^a = N_0^{-1} F_0^a$

$$\delta E^{(2)} = \frac{1}{2} N_0^{-1} F_0^a \sum (\sigma\sigma') \delta n_\sigma \delta n_{\sigma'} = \frac{1}{2} N_0^{-1} F_0^a (\bar{\sigma}_z)^2$$

Compare bare $\delta E = -\chi_0 B M_z$ renormalized $\delta E = -\chi_s B M_z = -\chi_0 B M_z + \underbrace{+\frac{1}{2} \frac{M_z^2 F_0^a}{\mu_B^2 N_0}}$

$M_z = \mu_B \sigma_z$

contribution of molecular field $B_{mol} = -\frac{\partial E}{\partial M_z} = -\frac{M_z F_0^a}{\mu_B^2 N_0}$

Then $M_z \stackrel{!}{=} -\frac{\partial E}{\partial B_{ext}} = \chi_0 B_{tot} = \chi_0 B_{ext} - \frac{\chi_0}{N_0} M_z F_0^a / \mu_B^2$

$$M_z \left(1 + \chi_0 F_0^a / \mu_B^2 N_0\right) = \chi_0 B_{ext}$$

Since $\frac{\chi_0}{N_0} = \mu_B^2 \rightarrow$ $\boxed{\chi = \chi_0 / \left(1 + \chi_0 F_0^a / \mu_B^2 N_0\right) = \frac{\chi_0}{1 + F_0^a}}$

4) Spin current $j_{i\alpha} = 2 \int \frac{d^3 p}{(2\pi)^3} \left(1 + F_1^a/3\right) \frac{p_i}{m^*} \delta n_{p\alpha}$

Spin effective mass $\frac{1}{m_s^*} = \frac{1}{m^*}\left(1 + F_1^a/3\right)$ $\boxed{\frac{m_s^*}{m} = \frac{1 + \frac{1}{3} F_1^s}{1 + \frac{1}{3} F_1^a}}$

In He^3 - typical Fermi-liquid $F_0^s = 10.8$ $F_0^a = -0.75$
compressibility reduced, spin-susceptibility is enhanced

Collective modes

Consider kinetic equation for quasiparticles

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial r} \frac{\partial \varepsilon}{\partial p} - \frac{\partial n}{\partial p} \frac{\partial \varepsilon}{\partial r} = 0$$

For the deviations from equilibrium

$$\frac{\partial \delta n}{\partial t} + \frac{\partial \delta n}{\partial r} \frac{\partial \varepsilon_0}{\partial p} - \frac{\partial \delta \varepsilon}{\partial r} \frac{\partial n_0}{\partial p} = 0$$ where $\delta \varepsilon$ is the Landau functional

$$\delta \varepsilon = \int f(p p') \delta n(p' r) \frac{d^3 p'}{(2\pi)^3} = 2 \Big\langle \underbrace{f(p_F, p_F') \nu_0}_{F(n, n')} \underbrace{\int d\xi_p' \delta n(\vec{n}' p', r)}_{u(n)} \Big\rangle$$

We assume $u(n) = \int \delta n(\vec{p})\, d\xi_p$
Physicaly $u(n)$ - is the displacement of Fermi surface

$$(\omega - k v_F) u(n) = k v_F \int F(n n') u(n') \frac{dn'}{4\pi}$$ - equation for eigenfrequencies $\omega(k)$

There are two types of solutions:
1) quasiparticle
2) collective

1) Quasiparticle solutions correspond to the singular solutions of the kinetic equation

$$u(n') = \delta(n-n') + u_{reg}(n') \Rightarrow$$ particle propagating in direction n

2) Collective modes correspond to the movements of Fermi liquid. They are called <u>zero sound</u>

Let us consider the simplest case when the interaction is local and isotropic, that is $F(n,n') = F_0$

Physical example of that is liquid He^3

In this case due to the symmetry $u = u(\theta)$

where θ is the angle between $\vec{n}$ and $\vec{k}$

$$(s - \cos\theta)\, u(\theta) = \frac{F_0}{2}\cos\theta \int u(\theta')\sin\theta' \frac{d\theta'}{\pi}$$ where $s = \frac{\omega}{V_F|k|}$

$$u(\theta) = \frac{A\cos\theta}{s - \cos\theta}$$

$$\frac{s}{2}\ln\frac{s+1}{s-1} - 1 = \frac{1}{F_0}$$

a) There is 1 solution for any $F_0 > 0$ (repulsive interaction) so that one mode with linear dispersion $\omega = S(F_0)V_F k$

b) Since $S(F_0) > 1$ this mode is <u>faster than any QP</u>

$$\omega > V_F k$$

Usually there is end point of the spectrum where it merges continuum

<u>Spin wave</u> $\quad f_{\sigma\sigma'} = F_0^a(\sigma\sigma') \quad u_a = u_1 - u_{-1}$

$$\delta\varepsilon_\sigma = F_0^a\, \sigma \langle u_a \rangle$$

$$(s - \cos\theta)\, u = \frac{F_0^a}{2}\cos\theta \langle u \rangle$$ - the similar wave as 0 sound

If $F_0^a < 0$ - different regime called "paramagnon"

To consider this regime we should use RPA and calculate <u>dynamical</u> susceptibilities χ

Poles of χ give collective modes

Density and spin dynamic response (RPA)

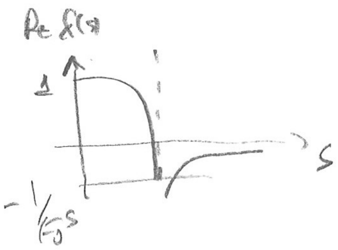
$$\chi_{dens}(q,\omega) = \frac{\chi_0(q,\omega)}{1+F_0^s\, \chi_0(q,\omega)/N_0}$$

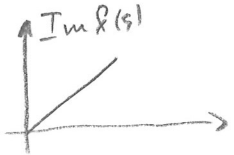
$$\chi_{spin}(q,\omega) = \frac{\chi_0}{1+F_0^a\, \chi_0/N_0}$$

Bare susceptibility is given by Lindhard function

$$\chi_0(q,\omega) = -\frac{2}{V}\sum_k \frac{n_{k+q}-n_k}{\omega+i0-(\varepsilon_{k+q}-\varepsilon_k)} = N_0\, f(s) \qquad s = v_F\, q/\omega$$

where $f(s) = 1 - \frac{s}{2}\ln\left|\frac{1+s}{1-s}\right| + i\frac{\pi}{2}\, s\, \Theta(1-|s|)$





$$\chi_{dens} = N_0 \frac{f(s)}{1+F_0^s f(s)}$$

$$\chi_{spin} = N_0 \frac{f(s)}{1+F_0^a f(s)}$$

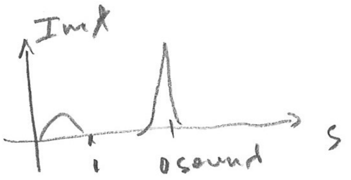
Collective modes as poles

1. Zero sound $\quad \mathrm{Im}\, f(s) = 0$
$\qquad 1+F_0^s f(s) = 0$

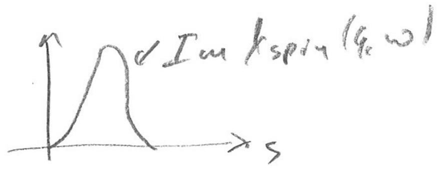
Solution for $F_0^s > 0$ is $s > 1$

Imaginary part of response function

$$\chi_{dens} = N_0 \frac{\mathrm{Re}\, f + i\, \mathrm{Im}\, f}{(1+F_0^s\, \mathrm{Re}\, f)^2 + i F_0^s\, \mathrm{Im}\, f} = N_0 \frac{\mathrm{Re}\, f\,(1+F_0^s\, \mathrm{Re}\, f) + (\mathrm{Im}\, f)^2 F_0^s - i\, \mathrm{Im}\, f}{(1+F_0^s\, \mathrm{Re}\, f)^2 + (F_0^s\, \mathrm{Im}\, f)^2}$$



If $F_0^a < 0$ imaginary part is enhanced.



$$\left(\frac{1}{i(p_n+q_n)-\varepsilon_{p+q}}\;\frac{1}{ip_n-\varepsilon_p}\right)\frac{1}{i(k_n-q_n)-\varepsilon_{k-q}}$$

$$T\sum_n \frac{1}{i(p_n+q_n)-\varepsilon_{p+q}}\;\frac{1}{ip_n-\varepsilon_p} = T\,\frac{1}{\varepsilon_{p+q}-\varepsilon_p-iq_n}\sum_{p_n}\left(\frac{1}{i(p_n+q_n)-\varepsilon_{p+q}} - \frac{1}{ip_n-\varepsilon_p}\right) = \frac{n(\varepsilon_{p+q})-n(\varepsilon_p)}{\varepsilon_{p+q}-\varepsilon_p-iq_n}$$

$$T\sum \frac{1}{\varepsilon_{p+q}-\varepsilon_p-iq_n}\;\frac{1}{i(k_n-q_n)-\varepsilon_{k-q}} = T\sum \frac{1}{iq_n+\varepsilon_p-\varepsilon_{p+q}}\;\frac{1}{i(q_n-k_n)+\varepsilon_{k-q}} =$$

$$= T\,\frac{1}{\varepsilon_{p+q}-\varepsilon_p+\varepsilon_{k-q}-ik_n}\sum\left(\frac{1}{iq_n+\varepsilon_p-\varepsilon_{p+q}} - \frac{1}{i(q_n-k_n)+\varepsilon_{k-q}}\right) = \frac{1}{\varepsilon_{p+q}-\varepsilon_p+\varepsilon_{k-q}-ik_n}\left(n(\varepsilon_{p+q}-\varepsilon_p)-n(-\varepsilon_{k-q})\right)$$

$$\Rightarrow \frac{\left(n(\varepsilon_{p+q})-n(\varepsilon_p)\right)\left(n(\varepsilon_{p+q}-\varepsilon_p)+1-n(\varepsilon_{k-q})\right)}{\varepsilon_{p+q}+\varepsilon_{k-q}-\varepsilon_p-ik_n}$$

Real-time - let's close all contours in lower 1/2 plane

$$\left(\frac{1}{E_p+E_q-\varepsilon_{p+q}+i\,\mathrm{sgn}(\varepsilon_{p+q})}\;\frac{1}{E_p-\varepsilon_p+i\,\mathrm{sgn}\,\varepsilon_p}\right)\frac{1}{\omega-E_q-\varepsilon_{k-q}+i\,\mathrm{sgn}(\varepsilon_{k-q})}$$

$$\int\frac{dE_p}{2\pi}\,\frac{1}{E_p+E_q-\varepsilon_{p+q}+i\,\mathrm{sgn}(\varepsilon_{p+q})}\;\frac{1}{E_p-\varepsilon_p+i\,\mathrm{sgn}\,\varepsilon_p} = \frac{1}{\varepsilon_{p+q}-\varepsilon_p-E_q+i(\mathrm{sgn}\,\varepsilon_p-\mathrm{sgn}\,\varepsilon_{p+q})}\left(\frac{1}{E_p+E_q-\varepsilon_{p+q}+i\,\mathrm{sgn}\,\varepsilon_{p+q}}-\frac{1}{E_p-\varepsilon_p+i\,\mathrm{sgn}\,\varepsilon_p}\right)$$

$$= \frac{\Theta(\varepsilon_{p+q})-\Theta(\varepsilon_p)}{\varepsilon_{p+q}-\varepsilon_p-E_q+i(\mathrm{sgn}\,\varepsilon_p-\mathrm{sgn}\,\varepsilon_{p+q})} \equiv \frac{\Theta(\varepsilon_{p+q})-\Theta(\varepsilon_p)}{\varepsilon_{p+q}-\varepsilon_p-E_q+i\,\mathrm{sgn}(\varepsilon_p-\varepsilon_{p+q})}$$

$\varepsilon_{p+q}>0 \quad \varepsilon_p>0 \Rightarrow 0$

$\varepsilon_{p+q}>0 \quad \varepsilon_p<0$
$\mathrm{sgn}(\varepsilon_p-\varepsilon_{p+q})=-1$
$\mathrm{sgn}\,\varepsilon_p-\mathrm{sgn}\,\varepsilon_{p+q}=-2$

$\varepsilon_{p+q}<0 \quad \varepsilon_p>0$
$\mathrm{sgn}(\varepsilon_p-\varepsilon_{p+q})=1$
$\mathrm{sgn}\,\varepsilon_p-\mathrm{sgn}\,\varepsilon_{p+q}=2$

$$\int\frac{dE_q}{2\pi}\,\frac{1}{\varepsilon_{p+q}-\varepsilon_p-E_q+i\,\mathrm{sgn}(\varepsilon_p-\varepsilon_{p+q})}\;\frac{1}{\omega-E_q-\varepsilon_{k-q}+i\,\mathrm{sgn}(\varepsilon_{k-q})} =$$

$$= \frac{1}{\omega-\varepsilon_{k-q}+\varepsilon_p-\varepsilon_{p+q}+i\,\mathrm{sgn}(\varepsilon_{k-q})-i\,\mathrm{sgn}(\varepsilon_p-\varepsilon_{p+q})}\left(\frac{1}{\varepsilon_{p+q}-\varepsilon_p-E_q+i\,\mathrm{sgn}(\varepsilon_p-\varepsilon_{p+q})}-\frac{1}{\omega-E_q-\varepsilon_{k-q}+i\,\mathrm{sgn}(\varepsilon_{k-q})}\right)$$

$$\equiv \frac{1}{\omega-\varepsilon_{k-q}+\varepsilon_p-\varepsilon_{p+q}+i\,\mathrm{sgn}(\varepsilon_{k-q}-\varepsilon_p+\varepsilon_{p+q})}\left(\Theta(\varepsilon_{p+q}-\varepsilon_p)-\Theta(-\varepsilon_{k-q})\right)$$

$$\Rightarrow \frac{1}{\omega-\varepsilon_{k-q}+\varepsilon_p-\varepsilon_{p+q}+i\,\mathrm{sgn}\,\omega}\left(\Theta(\varepsilon_{p+q})-\Theta(\varepsilon_p)\right)\left(\Theta(\varepsilon_{p+q}-\varepsilon_p)+1-\Theta(\varepsilon_{k-q})\right)$$