EXERCISE SET 1
PARTIAL DIFFERENTIAL EQUATIONS 2, 2019
EXERCISES ON 17.1.2019 AT 10.15 IN MAD380

1. Consider the equation $\Delta u(x) + \operatorname{div}(b(x)u(x)) = f(x)$ in $B(0,1)$, where $f : B(0,1) \to \mathbb{R}$ and $b : B(0,1) \to \mathbb{R}^n$ are given. Give a physical meaning for each of the terms along with a justification.
2. Let $u(x) = |x|^\alpha, x \in \mathbb{R}^n, x \neq 0$, where $\alpha \in \mathbb{R}$. Calculate $\Delta u(x)$ and find α so that $\Delta u(x) = 0$ in the classical sense.
3. Let $x \in \mathbb{R}^n,\ t > 0$, and
$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$
Show that $u_t(x,t) = \Delta u(x,t)$ i.e. u solves the heat equation in the classical sense.
4. Find a weak derivative (and show that it is a weak derivative) of
$$u : (-1,1) \to \mathbb{R}, \quad u(x) = \begin{cases} 0, & -1 < x \leq 0 \\ \sqrt{x}, & 0 < x < 1. \end{cases}$$
5. (a) Show that $\int_{\mathbb{R}} u\varphi' \, dx = \varphi(0)$ for every $\varphi \in C_0^\infty(\mathbb{R})$, where
$$u : \mathbb{R} \to \mathbb{R}, \quad u(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0. \end{cases}$$
   (b) Show that $u \notin W^{1,1}_{\text{loc}}(\mathbb{R})$
6. (a) Show that for $a, b \geq 0$ it holds that $(a+b)^p \leq C_1(a^p + b^p)$ and $a^p + b^p \leq C_2(a+b)^p$ for $p > 0$ where C_1, C_2 are independent of a, b.
   (b) Show that $||u||_{W^{1,p}(\Omega)}$ is equivalent with the norm
$$\sum_{|\alpha| \leq 1} \Big(\int_\Omega |D^\alpha u|^p \, dx \Big)^{1/p} \quad \text{if } 1 \leq p < \infty.$$