EXERCISE SET 3
PARTIAL DIFFERENTIAL EQUATIONS 2, 2019
EXERCISES ON TUESDAY 10.15-12, MAA203

1. Let $g \in W^{1,2}(\Omega)$. Show that
$$F(v) := \int_\Omega Dg \cdot Dv \, dx$$
is a bounded functional in $\hat{W}_0^{1,2}(\Omega)$.
2. Let $g \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$. Explain how you would prove existence for a weak solution of
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$
3. Let $u_k \rightharpoonup u$ converges weakly in $L^2(\Omega)$ as $k \to \infty$. Show that
$$||u||_{L^2(\Omega)} \leq \liminf_k ||u_k||_{L^2(\Omega)} .$$
4. Let $p > 2$. Formulate and prove the Dirichlet principle for the variational integral
$$I(v) = \frac{1}{p} \int_\Omega |Dv|^p \, dx.$$
5. Give a counterexample showing that
$$a_{ij} \in L^\infty(\Omega), b_i \in L^\infty(\Omega), c \in L^\infty(\Omega)$$
and
$$f \in L^2(\Omega)$$
is insufficient for weak solution u to be in $W^{2,2}_{\text{loc}}(\Omega)$. Also control that Example 3.5 in the lecture note is consistent with Thm 3.32.
6. Prove uniqueness for weak solutions to
$$-\Delta u(x) + \sum_{i=1}^n b_i(x) D_i u(x) + c(x)u(x) = f(x)$$
with standard assumptions on the data and the coefficients and for large enough c_0 for which $c \geq c_0$ (we proved this without $+\sum_{i=1}^n b_i(x) D_i u(x)$ in the lectures). Tips: Choose large enough c_0 such that $c \geq c_0$ in order to absorb the bad terms.