EXERCISE SET 2
PARTIAL DIFFERENTIAL EQUATIONS 2, 2019
EXERCISES ON TUESDAY 10-12, MAA203

1. Let $(0,2)=\Omega$, $f=1$, $b=0=c$,
$$a(x)=\begin{cases}1, & x\in(0,1]\\ 2, & x\in(1,2).\end{cases}$$
Consider the problem
$$\begin{cases}Lu=f, & x\in\Omega\\ u(0)=0=u(2).\end{cases}$$
Show that
$$u(x)=\begin{cases}-\frac{x^2}{2}+\frac{5}{6}x & x\in[0,1]\\ -\frac{x^2}{4}+\frac{5}{12}x+\frac{1}{6}, & x\in(1,2].\end{cases}$$
is a weak solution to the above problem.
2. Consider $\Omega=(0,2)$, $c=0=b$, $f=1$ and
$$a(x)=\begin{cases}x & x\in(0,1]\\ 1 & x\in(1,2)\end{cases}$$
and a problem
$$\begin{cases}Lu=f, & x\in\Omega\\ u(0)=0=u(2).\end{cases}$$
Show that
$$u(x)=\begin{cases}-x, & x\in(0,1]\\ -\frac{1}{2}x^2+2.5x-3, & x\in(1,2)\end{cases}$$
is not a weak solution.
3. Same data as in the previous exercise except
$$a(x)=\begin{cases}1+x & x\in(0,1]\\ 2 & x\in(1,2)\end{cases}$$
Find a weak solution and show that your function is a weak solution.
4. Consider a slight variant of the example in the lectures. Let $\alpha\in(0,1)$
$$\mathcal{A}=\begin{bmatrix}\frac{(1+a)x_1^2+x_2^2}{|x|^2} & a\frac{x_1x_2}{|x|^2}\\ a\frac{x_1x_2}{|x|^2} & \frac{x_1^2+(1+a)x_2^2}{|x|^2}\end{bmatrix}$$

where

$$a = \frac{\alpha(2-\alpha)}{(1-\alpha)^2}.$$

Show that

$$|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \leq (1+a)\,|\xi|^2$$

and that the a_{ij} are bounded.

5. Show that

$$u : B(0,1) \to \mathbb{R}, \quad u(x) = |x|^{-\alpha}\, x_1$$

with $x = (x_1, x_2)$ is in $W^{1,2}(B(0,1))$.

6. [1]Show that u in the previous exercise satisfies the weak formulation of the PDE, i.e. combining this with the previous exercise show that u is a weak solution.

[1]You may take for granted that u is a classical solution to $\operatorname{div}(\mathcal{A}(x)Du(x)) = 0$ when $x \neq 0$.