EXERCISE SET 4
PARTIAL DIFFERENTIAL EQUATIONS 2, 2019
EXERCISES ON TUESDAY 10.15-12, MAA203

1. Let $a_{ij}, b_i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $u \in W^{1,2}(\Omega)$ be a weak solution to $Lu = f$ in Ω. Show that, for any $\Omega' \Subset \Omega$,
$$\int_{\Omega'} |Du|^2 \, dx \le C \int_\Omega (u^2 + f^2) \, dx.$$
(Use $v = \eta^2 u$ as test function, where η is a suitable cut-off function).

2. Let $f \in L^2(\Omega)$, $g \in W^{1,2}(\Omega)$ and $u \in W^{1,2}(\Omega)$ be a weak solution to $\Delta u = f$ such that $u - g \in W_0^{1,2}(\Omega)$. Show that
$$||u||_{W^{1,2}(\Omega)} \le C(||g||_{W^{1,2}(\Omega)} + ||f||_{L^2(\Omega)}).$$

3. Let $u \in W_{\text{loc}}^{1,2}(\Omega)$ be a weak subsolution[1] to $\Delta u = 0$ in Ω. Let $0 < r < R < \infty$ such that $B(x_0, R) \subset \Omega$ and consider a cut-off function $\eta \in C^\infty(B(x_0, R))$ such that $0 \le \eta \le 1$, $\eta = 1$ in $B(x_0, r)$ and $|D\eta| \le C/(R - r)$. Show
$$\int_{B(x_0,R)} \eta^2 \left|D(u-k)_+\right|^2 \, dx \le \Big(\frac{C}{R-r}\Big)^2 \int_{B(x_0,R)} \left|(u-k)_+\right|^2 \, dx,$$
where $k \in \mathbb{R}$ and $u_+ = \max\{u, 0\}$.

4. Does the ess sup-estimate
$$\operatorname*{ess\,sup}_{B(x_0,r/2)} u \le k_0 + C\Big(\fint_{B(x_0,r)} |(u-k_0)_+|^2 \, dx\Big)^{1/2}$$
hold for a subsolution to $\Delta u = 0$?

5. Let $n > 2$, $u : \mathbb{R}^n \to (0, \infty]$ be the function defined by $u(x) = |x|^{2-n}$ and $k > 0$. Show that $\min\{u, k\}$ is a weak supersolution to $\Delta u = 0$.

6. Show by a counterexample that the esssup-estimate in Exercise 4 does not hold in the case of a supersolution.

[1] A function $u \in W_{\text{loc}}^{1,2}(\Omega)$ is a weak subsolution (supersolution) to $\Delta u = 0$ if
$$\int_\Omega Du \cdot D\varphi \, dx \le (\ge)\, 0$$
for every *nonnegative* $\varphi \in C_0^\infty(\Omega)$.