EXERCISE SET 5
PARTIAL DIFFERENTIAL EQUATIONS 2, 2019
EXERCISES ON TUESDAY 10.15-12, MAA204

1. Let $k \in \mathbb{R}$ and define $A(k,r) = B(x_0,r) \cap \{x \in \Omega : u(x) > k\}$ and $k_0 = (m(2r) + M(2r))/2$. The measure decay lemma in the De Giorgi's method states that if u is a weak solution to $\Delta u = 0$ and
$$|A(k_0,r)| \leq \gamma \, |B(x_0,r)|\,, \quad 0 < \gamma < 1,$$
then
$$\lim_{k \nearrow M(2r)} |A(k,r)| = 0.$$
Is this result still true if we just assume u to be a weak subsolution?

2. Let $n > 2$, $u : \mathbb{R}^n \to (0,\infty]$ be the function defined by $u(x) = |x|^{2-n}$ and $k > 0$. Show that $\min\{u,k\}$ is a weak supersolution to $\Delta u = 0$ that does not satisfy the measure decay lemma.

3. Let $\operatorname*{osc}_{B(x_0,2r)} u = M(2r) - m(2r)$. Assume that u satisfies
$$\operatorname*{osc}_{B(x_0,r/4)} \leq \gamma \operatorname*{osc}_{B(x_0,r)} u \tag{$*$}$$
for some $\gamma \in (0,1)$. Show that there exist constants $C > 0$ and $\alpha \in (0,1)$ such that
$$\operatorname*{osc}_{B(x_0,r/4)} u \leq C\Big(\frac{r}{R}\Big)^{\alpha} \operatorname*{osc}_{B(x_0,R)} u$$
for every $R \geq r$. (Hint: choose $j \in \mathbb{N}$ such that $4^{j-1}r \leq R < 4^j r$ and iterate inequality $(*)$ j times).

4. Let $u : \Omega \to \mathbb{R}$ be a function satisfying $(*)$ with $\gamma = \frac{1}{2}$ and assume that $|u|$ is bounded in $B(0,1) \subset \Omega$. Show that there exist $C > 0$ and $\alpha \in (0,1)$ such that
$$|u(x) - u(y)| \leq C|x-y|^{\alpha}$$
for every $x, y \in B(0,\frac{1}{8})$. What is the value of α? (Hint: assume that $x \neq y$, choose $\frac{r}{4} = |x-y|$, $x_0 = (x+y)/2$ and apply the inequality obtained in the previous exercise).

5. Does Hölder continuity hold for subsolutions of $\Delta u = 0$?