EXERCISE SET 6
PARTIAL DIFFERENTIAL EQUATIONS 2, 2019
EXERCISES ON TUESDAY 10.15-12, MAA203

1. Consider the elliptic equation
$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + cu = 0 \qquad (*)$$
whit $c \geq 0$. Does the weak maximum principle hold for weak subsolutions to $(*)$? And for supersolutions? Provide an explanation or a counterexample in each case.

2. Formulate and prove the weak minimum principle for weak solutions $(*)$ to.

3. Combine the weak maximum principle and the weak minimum principle to show that
$$\operatorname*{ess\,sup}_{\Omega} |u| \leq \sup_{\partial\Omega} |u|$$
for every $u \in W^{1,2}(\Omega)$ weak solution to $(*)$. Does this inequality hold for subsolutions or supersolutions?

4. Give a counterexample to the weak maximum principle for $(*)$ if we drop the assumption $c \geq 0$.

5. Give a counterexample to the weak maximum principle for $(*)$ if we replace $\sup_{\partial\Omega} u_+$ by $\sup_{\partial\Omega} u$.

6. Give an example showing that the Hopf lemma does not hold if we drop the interior ball condition. (Hint: consider the harmonic function $u(x, y)$ defined as the real part of the complex function $f(z) = z^\alpha$ and choose $\alpha > 0$ and $\Omega \subset \mathbb{R}^2$).